* Relationship between indep. and prod. msres.

Recall. $X_1, \dots X_n$ rvs on $(\Omega, \mathcal{F}, P)$ are indep. if $\forall B_1, \dots B_n$ Borel sets

$P(\omega: X_1(\omega) \in B_1, X_2(\omega) \in B_2 \dots X_n(\omega) \in B_n)$

$\overset{\text{def}}{=} P(\bigcap_{i=1}^{n} X_i^{-1}(B_i)) \overset{*}{=} \prod_{i=1}^{n} P(X_i^{-1}(B_i))$

$\overset{\text{def}}{=} \prod_{i=1}^{n} P(\omega: X_i(\omega) \in B_i)$ $\quad (\Omega, \mathcal{F}, P)$

Recall The Law of a RV X is the prob. msre on $(\mathbb{R}, \mathcal{B})$ $\mu_X(B) := P(X^{-1}(B))$

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new. Def: The joint Law or dist of rvs. $X_1 \dots X_n$ defined on $(\Omega, \mathcal{F}, P)$ is a pm on $(\mathbb{R}^n, \mathcal{B}^n)$, $\mu_{X_1 \dots X_n}$

$\mu_{X_1 \dots X_n}(B) := P(\omega: (X_1(\omega) \dots X_n(\omega)) \in B)$ $B \in \mathcal{B}^n$.

Exercise $X_1, \dots X_n$ are indep iff $\mu_{X_1, \dots, X_n} = \mu_{X_1} \times \mu_{X_2} \dots \times \mu_{X_n}$

$\mu_1 \times \mu_2 \times \dots \times \mu_n \, (X_1 \times X_2 \dots \times X_n)$

$= \prod_{i=1}^{n} \mu(X_i)$. In order to keep things simple. assume $\mu(X_i) = 1$ $\forall i$

Inf. product msre of Leb msre on $[0,1]$.

$([0,1]^n, \mathcal{B}^n, \underbrace{m \times \dots \times m}_{n})$. well know.

move to inf. prod

$X = [0,1]^{\mathbb{N}}$ $\mathbb{N}$ pos. integers

$= \{ (a_1, a_2, a_3 \ldots) : a_i \in [0,1] \ \forall i$.

Let $\mathcal{B}^\infty$ be the smallest σ-alg containing sets of the form

$$\underline{B_1 \times B_2 \ldots \times B_n \times [0,1] \times [0,1] \times [0,1] \ldots}$$

each $B_i \in \mathcal{B}_1$ $= \{ (a_1, a_2, \ldots) : a_1 \in B_1, a_2 \in B_2 \ldots a_n \in B_n \}$

Let m Leb. mare on $[0,1]$.

Thm: $\exists$ a p.m. m^∞ on $(X, \mathcal{B}^\infty)$ s.t. $\forall n \ \forall B_1, \ldots B_n \in \mathcal{B}_1$,

$$m^\infty (B_1 \times B_2 \times \ldots \times B_n \times [0,1] \times [0,1] \ldots)$$

$$= m(B_1) m(B_2) \ldots m(B_n).$$

Remarks:

(1) one can do unctble products.

(2) versions of this for "non product things" – Kol. extension theorem.

(3) can be techniqual; need some topological assumptions in general.

Easier proof

Thm. On $([0,1]^{\mathbb{N}}, \mathcal{B}^{\infty}, m^{\infty})$

$\exists$ RV's $X_1\ X_2 \ldots$ s.t.

Law of X_i is m $\forall i$, X_i's are indep.

Pf. Let $X_i : [0,1]^{\mathbb{N}} \longrightarrow [0,1]$ be

$X_i((a_1 a_2 \ldots\ldots)) = a_i$ "proj. onto i^{th} coord."

(1) X_i is a RV i.e. msble

$\forall B \in \mathcal{B}_1$, $X_i^{-1}(B) = \{(a_1 \ldots) : X_i(a_1 \ldots) \in B\}$

$= \{(a_1 \ldots) : a_i \in B\}$

$= [0,1]\times[0,1] \ldots \times \underset{i-1}{[0,1]} \times \underset{i}{B} \times \underset{i+1}{[0,1]} \ldots$

$\in \mathcal{B}^{\infty}$.

(2) Law of X_i $\quad B \in \mathcal{B}_1$

$\mu_{X_i}(B) = P(X_i^{-1}(B)) =$

$m^{\infty}[\,[0,1]\times \ldots [0,1]\ \underset{i}{B} \times [0,1] \ldots)$

$= m(B)$

(3) Indep. NTS

$\mu_{X_1, \ldots, X_n} = \mu_{X_1} \times \ldots \times \mu_{X_n}$

NTS $\mu_{X_1 \ldots X_n}(B_1 \times B_2 \ldots \times B_n) = \mu_{X_1} \times \ldots \times \mu_{X_n}(B_1 \ldots B_n)$

RHS $= \prod \mu_{X_i}(B_i) = \prod_{i=1}^{n} m(B_i)$

LHS $= m^{\infty}((a_1 \ldots) : X_1 \ldots X_n (a_1 \ldots) \in B_1 \times \ldots B_n)$

$= m^{\infty}(B_1 \times B_2 \ldots \times B_n \times [0,1] \times [0,1] \ldots)$

$= \prod_{i=1}^{n} m(B_i)$. $\square$

Construct an inf. # of uniform RVs

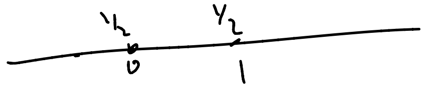
€. what if we want to construct indep. RV's $X_1 \ldots$ $\quad P(X_i) = \begin{cases} 1 & 1/2 \\ 0 & 1/2 \end{cases}$.

Take our X_1 X_2 ——.

$$Y_1 = \begin{cases} 1 & X_1 \in [0, 1/2] \\ 0 & X_1 \in [1/2, 1] \end{cases}$$

Exercise. Show Y_i's are RV's, indep and their Law

$$\frac{\delta_1 + \delta_0}{2}$$



If $Y_1 \ldots$ are iid, $\exists f : [0,1] \to \mathbb{R}$ s.t $Y_1 = f(X_1)$ works.

Lying. $f(x) \neq F_Y^{-1}(x)$

Construct μ^∞ on $([0,1]^{\mathbb{N}}, \mathcal{B}^\infty)$ by applying prev. theorem.

Let $X = [0,1]^{\mathbb{N}}$. Let $\mathcal{F}_n$ be the σ-alg generated by sets of the form $\{B_1 \times B_2 \times \cdots \times B_n \times [0,1] \times [0,1] \cdots\}$, $B_i \in \mathcal{B}$

$\boxed{\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3} \cdots$

Let $\mathcal{Q} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

L1. $\mathcal{Q}$ is an alg.

Pf. Let $A, B \in \mathcal{Q}$. $\Rightarrow A \in \mathcal{F}_n$ some n. $B \in \mathcal{F}_m$ .. Let $N = \max\{n, m\}$. comp. exercise

$A, B \in \mathcal{F}_N \Rightarrow A \cup B \in \mathcal{F}_N \subseteq \mathcal{Q}$..

L.2 $\mathcal{Q}$ is not a σ-alg.

Let $A_n = \underbrace{[0,\tfrac12] \times [0,\tfrac12] \cdots \times [0,\tfrac12]}_{n} \times [0,1] \cdots$

$A_n \in \mathcal{F}_n \subseteq \mathcal{Q}$.

$\bigcap_{n=1}^{\infty} A_n = [0,\tfrac12] \times [0,\tfrac12] \times \cdots \times [0,\tfrac12] \cdots$

$\nearrow$ not in $\mathcal{Q}$ since not in $\mathcal{F}_n$ for any n.

Define a pre msre on $\mathcal{Q}$

$\mu_0^\infty(B_1 \times B_2 \cdots \times B_n \times [0,1] \times [0,1] \cdots)$

$:= \prod_{i=1}^{n} m(B_i)$.

Just as for product msres, this extends to a p.m. on $\mathcal{F}_n$.

basically n-dim measure

Claim: μ_0^∞ is a premeasure on $\mathcal{C}$

If true, then μ_0^∞ extends to a p.m μ^∞ on $\sigma(\mathcal{C}) = \mathcal{B}^\infty$.

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must prove claim. $\mu_0^\infty(\emptyset) = 0$ ✓

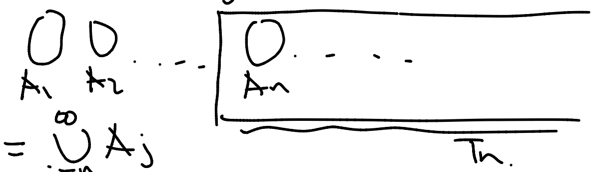
Let $\overbrace{A_1, A_2, \dots}^{\text{disjoint}} \in \mathcal{C}$ and $\cup A_i \in \mathcal{C}$,

we NTS $\mu_0^\infty(\cup A_i) = \sum_{i=1}^\infty \mu_0^\infty(A_i)$.

Remark. $\geq$ easy

$$\mu_0^\infty(\cup A_i) \geq \mu_0^\infty\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0^\infty(A_i) \quad \forall n$$

claim* $\mu_0^\infty\left(\bigcup_{j=n}^\infty A_j\right) \longrightarrow 0$ as $n \to \infty$



$T_n = \bigcup_{j=n}^\infty A_j$

Note $T_1 \supseteq T_2 \supseteq T_3 \dots$ and note

$\bigcap_{n=1}^\infty T_n = \emptyset$ Exercise

$\Rightarrow \mu_0^\infty(T_n) \longrightarrow 0$

cont from above can't be used

claim* $\Rightarrow$ ~~pro~~. pre msre

$$\mu_0^\infty\left(\bigcup_{i=1}^\infty A_i\right) \underset{f.a.}{=} \mu_0^\infty\left(\bigcup_{j=1}^{n-1} A_j\right) + \mu_0^\infty(T_n)$$

$$\underset{fa}{=} \underbrace{\sum_{j=1}^{n} \mu_0^\infty\left(A_j\right)}_{} + \mu_0^\infty(T_n)$$

Let $n \to \infty$ $\quad \sum_{j=1}^{\infty} \ldots \quad$ $\mu_0^\infty(T_n) \to 0$

claim* $T_1 \supseteq T_2 \supseteq \ldots T_n$.

$\bigcap_{n\geq 1}^{\infty} T_n = \emptyset \Rightarrow \mu_0^\infty(T_n) \longrightarrow 0$.

Recall compactness, in $[0,1]$

~~$B_1, B_2, \ldots$~~ closed sets. $B_1 \neq \emptyset$.

$B_1 \supseteq B_2 \supseteq \quad \Rightarrow \bigcap_{i=1}^{\infty} B_i \neq \emptyset$

~~$(1/2, 1/n)$~~ $B_n = (1/2, \frac{1}{2} + 1/n)$. shows need closed

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also true in $[0,1]^{\mathbb{N}}$.

~~$T_1 \supseteq T_2$~~ $\supseteq$. assume $\mu_0^\infty(T_n) \searrow \varepsilon_0 > 0$.

Note $T_n \neq \emptyset$, $T_1 \supseteq T_2 \supseteq \ldots$ $\bigcap_{n=1}^\infty T_n = \emptyset$.

could not happen if T_n's compact

Idea Estimate T_n from inside by a closed set C_n.

Find $C_n \subseteq T_n$ closed. s.t.

$$\underline{\mu_0^\infty(T_n \setminus C_n) < \frac{\varepsilon_0}{2^{n+1}}}$$

Let $\underline{B_n = \bigcap_{i=1}^n C_i}$ closed. $\underline{B_n} \subseteq C_n \subseteq \underline{T_n}$. $*$

Key pt. $B_n \neq \emptyset$ $**$

$B_1 \supseteq B_2 \supseteq B_3 \ldots$

$\overset{**}{\underset{\text{compactness}}{\Longrightarrow}} \bigcap_{n=1}^\infty B_n \neq \emptyset$

$\overset{}{\underset{*}{\Longrightarrow}} \bigcap_{n=1}^\infty T_n \neq \emptyset$. ↯

$**$ Show $\mu_0^\infty(B_n) > 0 \quad \forall n$.

$\mu_0^\infty(T_n \setminus B_n) = \mu_0^\infty(T_n \setminus \bigcap_{i=1}^n C_i)$

$\leq \mu_0^\infty\left(\bigcup_{i=1}^n (T_i \setminus C_i)\right)$

$\leq \sum_{i=1}^n \mu_0^\infty(T_i \setminus C_i)$

$\leq \sum_{i=1}^n \frac{\varepsilon_0}{2^i 2} \leq \frac{\varepsilon_0}{2}$

But $\mu_0^\infty(T_n) \geq \varepsilon_0 \Rightarrow \underline{\mu_0^\infty(B_n) > 0}$

$\square$