1. Remark on PS 34.

converse of Borel Cantelli is false

$\sum_{i=1}^{\infty} P(A_i) < \infty \Rightarrow P(\overline{\lim}\, A_i) = 0$

$\Leftarrow$ false

$([0,1], \mathcal{B}, m)$, Leb. msr. $A_i = (0, \frac{1}{i})$.

$\sum P(A_i) = \infty$ $\quad \overline{\lim}\, A_i = \emptyset$

A_i indep $\Rightarrow$ conv. true

(A_i) pairwise indep. $\Rightarrow$ converse true

PS44

1. $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$

$\supseteq$ trivial

$\subseteq$ $x \in LHS$ $\quad f(x) \in A,\ f(x) \in B$

$\Rightarrow f(x) \in A \cap B$

$x \in RHS$

2. $f(A) \cap f(B) \overset{?}{=} f(A \cap B)$

$\supseteq$ triv.

$\subseteq$ false

$f: \ a \mapsto 1, \ b \mapsto 1$

$f(A) = \{ f(x) : x \in A \}$

$A = \{a\}$ $\quad B = \{b\}$.

$LHS = \{1\}$

$RHS = \emptyset$.

3. ~~$f^{-1}(A) - f A$~~

$f^{-1}(A^c) = (f^{-1}(A))^c$ true ✓.

$f(A^c) \overset{?}{=} (f(A))^c$ false

$a \to z$, $b \to z$, w

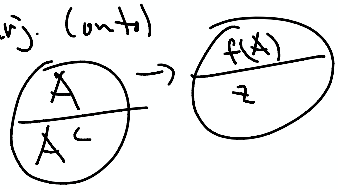
$A = \{a\}$

~~$B \to \{y\}$~~

LHS $= \{z\}$ RHS $z \{w\}$

3. $(f(A))^c \subseteq f(A^c)$ if f is surj. (onto)

$z \in f(A^c)$



57. Find $f_n \geq 0$ on $[0,1]$, $f_n \to 0$ p.w.

$\int f_n \to 0$ but $\nexists\, g$ $\ni$

$|f_n| \leq g$ and $\int g < \infty$

if $g \geq f_n$, then

$|g| \geq n$ on $[\frac{1}{n+1}, \frac{1}{n}]$

$\Rightarrow \int g = \sum_{n=1}^{\infty} n\, m(I_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

Gen. of the LDC

If $f_n \to f$ in measure

and $|f_n| \le g$ $\forall n$ $\forall x$,

and $\int |g| < \infty$,

then $\int f_n \to \int f$.

Hint: Fact $f_n \to f$ in measure, then $\exists$ a subseq. f_{n_k} s.t. $f_{n_k} \to f$ for a.e. x.

Note $|\int f_n| \le \int |g| < \infty$

$\{\int f_n\}$ are bdd.

Goal: $\int f_n \to \int f$.

assume not, then $\exists$ a subseq. $\int f_{n_k} \to L \neq \int f$.

$f_{n_k} \to f$ in msre

$\Rightarrow$ there is a further subseq. $f_{n_k'}$ s.t.

Hint

$f_{n_k'} \to f$ a.e. x

$\int f_{n_k'} \xrightarrow[\text{LDC}]{} \int f$

$\downarrow$

L

$\neq$

Related comment $[0,1)$

$x_n \to x$ iff every subseq. of x_n has a further subseq. conv. to x.

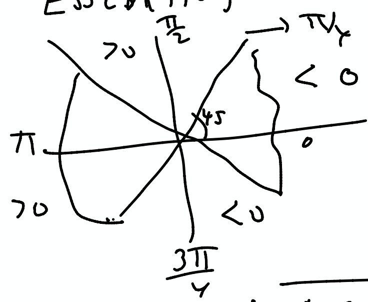
Exercise ~~Prove~~ Prove of

p1 #3. $([0,1], \mathcal{B})$

$\nu(A) = \int_A (\sin 2\pi x) - (\cos 2\pi x)\, dx$ $A \in \mathcal{B}$ ~~on~~ $[0,1)$

Find the ~~Jord~~ Hahn and Jordan decomp

Essentially where integrand pos, neg.



integrand is $\begin{cases} \geq 0 & [1/8, 5/8) \\ < 0 & [0, \frac{1}{8}] \cup (\frac{5}{8}, 1]. \end{cases}$

$P = [1/8, 5/8] \cup \{1/100, 1/100\}$

$N = [0, 1/8) \cup (5/8, 1]$

✓ $A \subseteq P \Rightarrow \nu(A) = \int_A f\, dx \geq 0.$

$A \subseteq N$ . . . ≤ 0

Hahn

Jordan $\nu^+(A) = \nu(A \cap P) = \int_{A \cap P} f(x)\, dx$

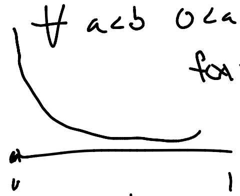
$\nu^-(A) = -\nu(A \cap N) = \int_{A \cap N} (-f(x))\, dx.$

$\nu = \nu^+ - \nu^-$

Find $f \ge 0$ on $[0,1)$ s.t.

$f(x) < \infty \ \forall x$, $\forall a<b$, $0<a<b<1$

$$\int_a^b f(x)\,dx = \infty$$



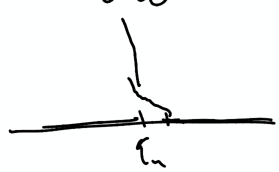
$f(x) = \begin{cases} \frac{1}{x} & x>0 \\ 0 & x=0 \end{cases}$ $\quad \int_0^1 f(x) = \infty$

Idea: find f "looks like $\frac{1}{x}$ everywhere"

Sol. $\mathbb{Q}$ = rational #s in $[0,1)$

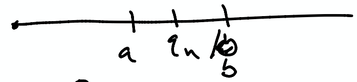
$q_1, q_2, q_3 \ldots$

$$f_n(x) = \begin{cases} \frac{1}{x-q_n} & x \in [q_n, q_n + \frac{1}{2^n}) \\ 0 & \text{o w} \end{cases}$$



Note $\int_0^1 f_n = \infty$

Note if $q_n \in (a,b)$ then $\int_a^b f_n = \infty$



Let $f = \sum_{n\ge 1}^{\infty} f_n$.

Claim $\forall a<b$, $\int_a^b f = \infty$.

pf Find $q_n \in (a,b)$.

$$\int_a^b f \ge \int_a^b f_n = \infty.$$

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maybe $f(x) = \infty \ \forall x$

Claim $f(x) < \infty$ a.e. x.

pf. BC. Let $A_n = [q_n, q_n + \frac{1}{2^n}]$

$\sum_n m(A_n) < \infty \Rightarrow m(\overline{\lim} A_n) = 0$.

$\Rightarrow$ a.e. x is in only finitely many A_n's

$\Rightarrow$ a.e. x only fin many f_n's are >0. $\Rightarrow f(x) < \infty$ a.e. x.

Now let $\tilde{f}(x) = \begin{cases} f(x) & f(x) < \infty \\ 0 & f(x) = \infty \end{cases}$

$\tilde{f}$ works. Since $\tilde{f} = f$ a.e. $\square$

Folland suggested problem:

one key goal is $\underset{\text{Leb. msbles}}{\mathcal{M}} \supsetneq \underset{\text{Borel sets}}{\mathcal{B}}$

Rem. one way is to show card. of $\mathcal{M}$ is $>$ card $(\mathcal{B})$

$f: \mathbb{R} \to \mathbb{R}$ is msble. can be

Leb msble $f: (\mathbb{R}, \mathcal{M}) \to \mathbb{R}$ msble or $\mathcal{B} \subsetneq \mathcal{M}$

$f \; (\mathbb{R}, \mathcal{B}) \xrightarrow[\text{Borel msble}]{} \mathbb{R}$ msb

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g Borel msble and f is Leb msre

then $g \circ f$ is ~~msble~~ Leb. msre

$\underset{\mathbb{R}}{\bigcirc} \xrightarrow{f} \underset{\mathbb{R}}{\bigcirc} \xrightarrow{g} \underset{\mathbb{R}}{\bigcirc}$

$(g \circ f)^{-1}(B) = \underbrace{f^{-1}(g^{-1}(B))}_{\in \mathcal{M}}$, $g^{-1}(B) \in \mathcal{B}$

$f \circ g$ ~~does not work~~ not necess Leb. msb

$\bigcirc \xrightarrow{g} \bigcirc \xrightarrow{f} \bigcirc$ OB

$f^{-1}(B) \in \mathcal{M} \not\Rightarrow g^{-1}(f^{-1}(B)) \in \mathcal{M}$

Ex 29. A ~~&~~ $\subseteq [0,1]$, $m(A)>0$,

A contains a nonmble set.

($m(A)=0 \Rightarrow A \not\supseteq$ nonmble set

since $B\subseteq A$, $m^*(B)=0 \Rightarrow B\in\mathfrak{M}$ → saw this earlier

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$f:[0,1]\to[0,1]$ standard cantor tern fcn.

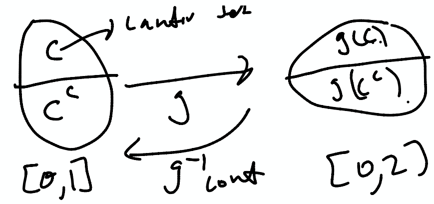
f cont. $f(0)=0$ $f(1)=1$ $x\le y \Rightarrow f(x)\le f(y)$

$g(x): f(x)+x$ on $[0,1]$.

$g(0)=0$ $g(1)=2$ g is cont. ~~bijection~~

g is strictly increasing $x<y \Rightarrow g(x)<g(y)$

g: strictly inc. cont bij from $[0,1]$ to $[0,2]$



claim $g(C)$ has measure 1.

Pf. Show $m(g(C^c))=1$

$C^c=\bigcup_i I_i$ disjoint open intervals

f is constant on each I_i

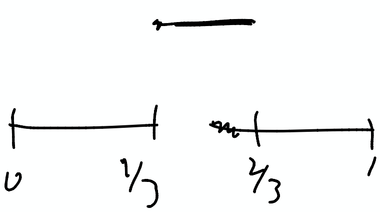
$\Rightarrow$ g has slope on $I_i = 1$

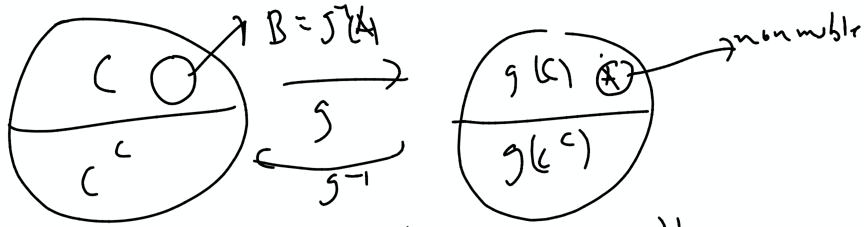
$m(g(I_i)) = m(I_i)$

$g(C^c) = g(\bigcup I_i) = \bigcup_i g(I_i)$

$\Rightarrow m(g(C^c)) = \sum m(g(I_i))$

$= \sum m(I_i) = 1$ $\square$





Choose $A \subseteq g(C)$ s.t. A is nonmsble

Let $B = g^{-1}(A)$. B winnen. $B \subseteq C \Rightarrow m^*(B) = 0 \Rightarrow B$ Leb. msble

B not Borel. Pf. g^{-1} cont map. $\Rightarrow g^{-1}$ Borel msble

If $\underline{B \text{ Borel}}$, then $(g^{-1})^{-1}(B) \in \mathcal{B}$; i.e. $A \in \mathcal{B}$. ↯ A ~~m~~ Leb. msble

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Exerc $\underbrace{F = I_B}_{\text{Leb. msble}}$ $\underbrace{G = g^{-1}}_{\text{Borel msble}}$ $F \circ G = I_A$ not Leb. msble

$\square$