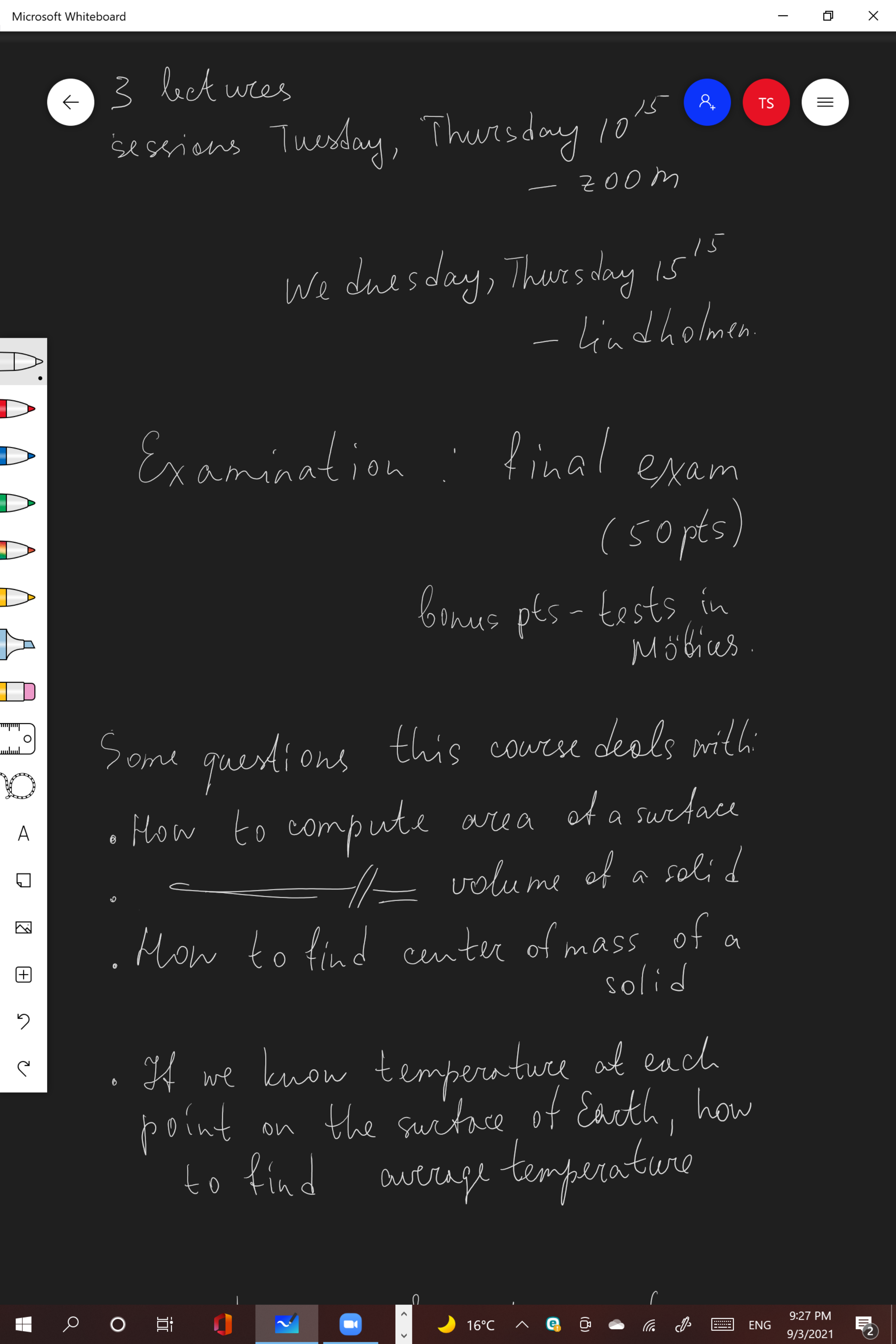


3 lectures

sessions Tuesday, Thursday 10^{15}
— zoom

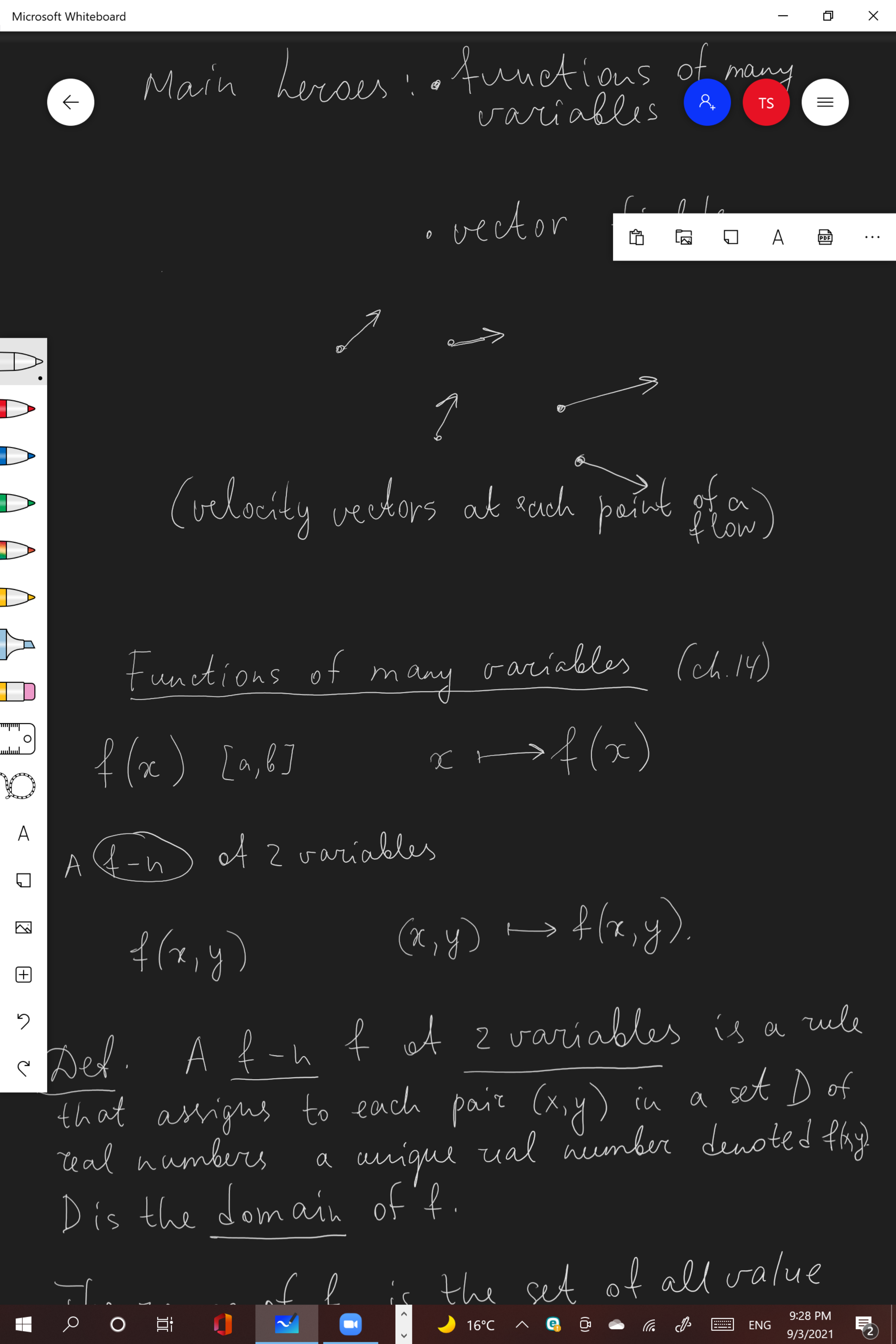
Wednesday, Thursday 15^{15}
— Lindholmen.

Examination: final exam
(50 pts)

bonus pts - tests in Möbius.

Some questions this course deals with:

- How to compute area of a surface
- ——— // ——— volume of a solid
- How to find center of mass of a solid
- If we know temperature at each point on the surface of Earth, how to find average temperature



Main heroes: • functions of many variables

• vector

(velocity vectors at each point of a flow)

Functions of many variables (ch. 14)

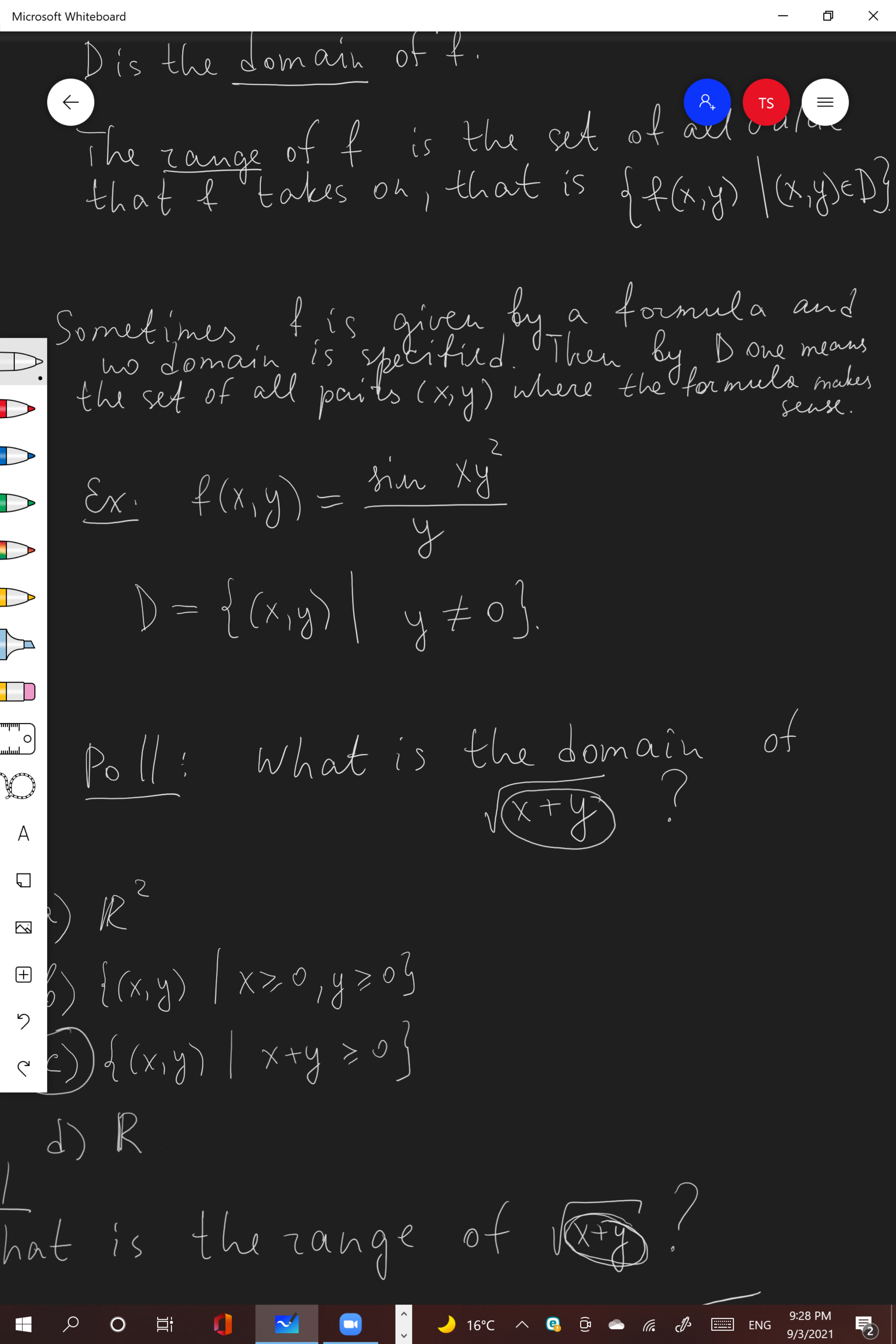
$f(x)$ $[a,b]$ $x \longmapsto f(x)$

A f-n of 2 variables

$f(x,y)$ $(x,y) \longmapsto f(x,y).$

Def. A f-n f of 2 variables is a rule that assigns to each pair (x,y) in a set D of real numbers a unique real number denoted $f(x,y)$.
D is the domain of f.

The range of f is the set of all value



D is the domain of f.

The range of f is the set of all values that f takes on, that is $\{f(x,y) \mid (x,y)\in D\}$.

Sometimes f is given by a formula and no domain is specified. Then by D one means the set of all pairs (x, y) where the formula makes sense.

Ex: $f(x,y) = \dfrac{\sin xy^2}{y}$

$D = \{(x,y) \mid y \neq 0\}$.

Poll: What is the domain of $\sqrt{x+y}$?

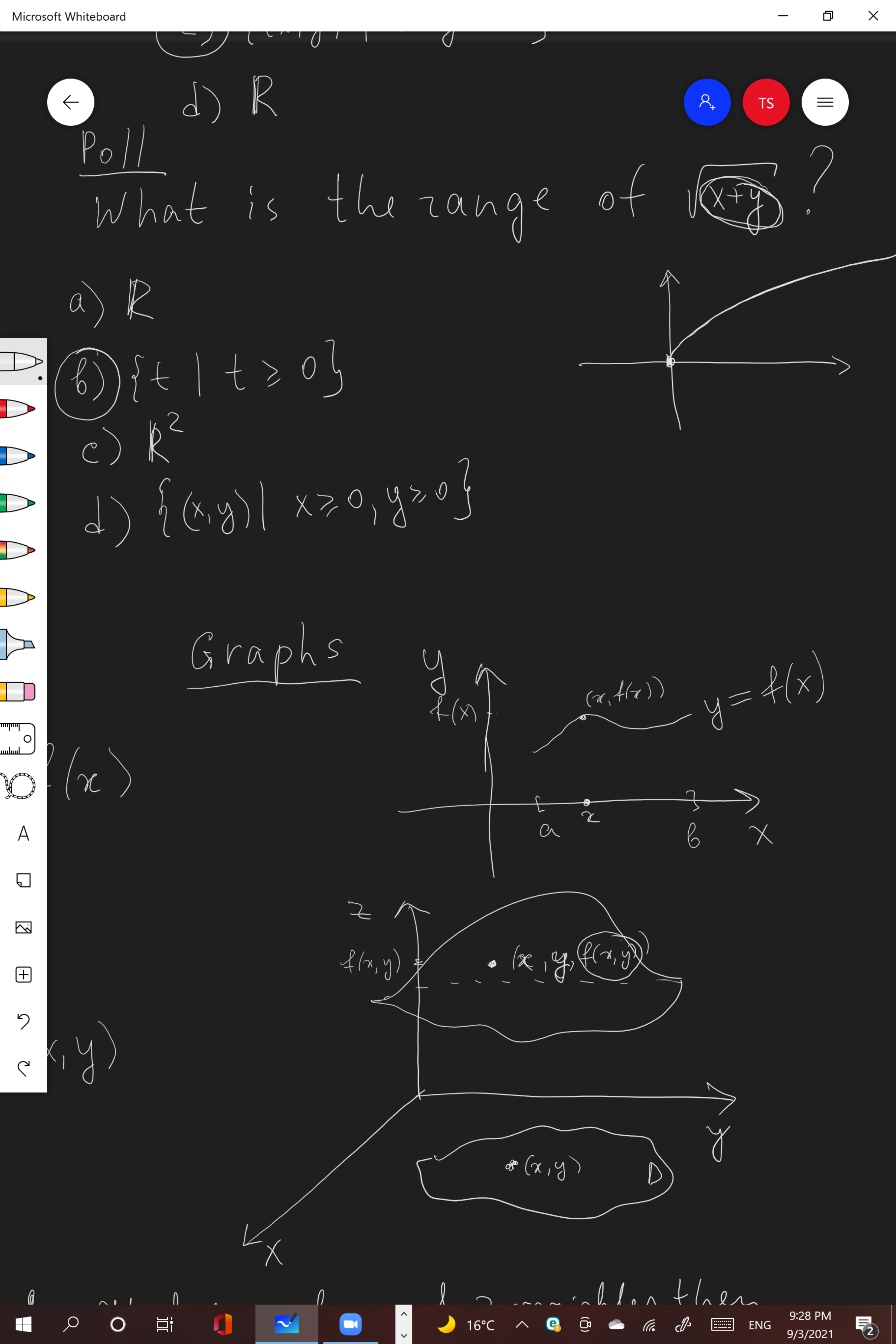
a) $\mathbb{R}^2$

b) $\{(x,y) \mid x \geq 0, y \geq 0\}$

c) $\{(x,y) \mid x+y \geq 0\}$

d) $\mathbb{R}$

hat is the range of $\sqrt{x+y}$?



d) $\mathbb{R}$

Poll

What is the range of $\sqrt{x+y}$?

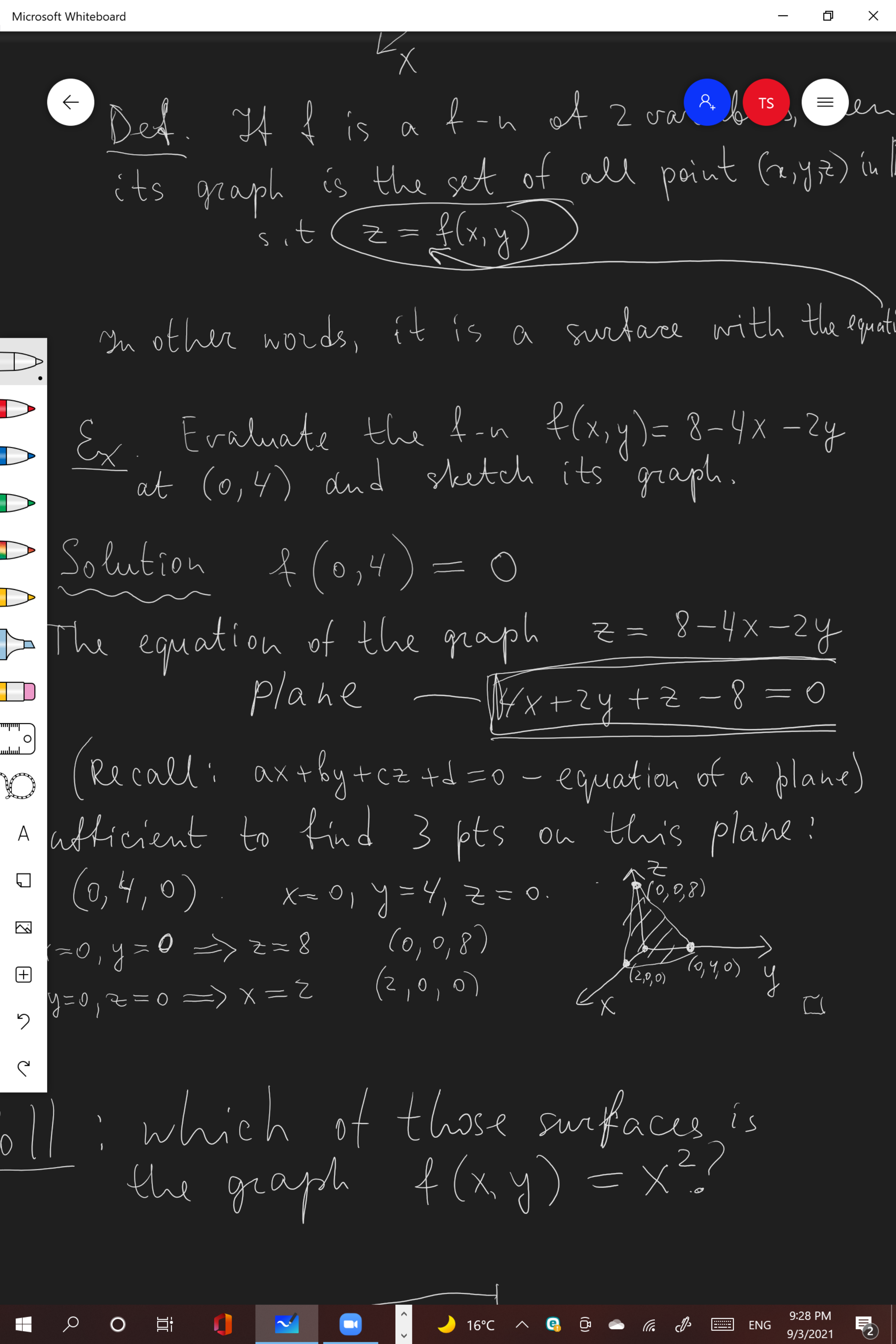
a) $\mathbb{R}$

[✓ b)] $\{t \mid t \geq 0\}$

c) $\mathbb{R}^2$

d) $\{(x,y) \mid x \geq 0, y \geq 0\}$

Graphs



X

Def. If f is a f-n of 2 va b s, en

its graph is the set of all point (x,y,z) in

s.t $\boxed{z = f(x,y)}$

In other words, it is a surface with the equati

Ex. Evaluate the f-n $f(x,y) = 8-4x-2y$ at $(0,4)$ and sketch its graph.

Solution $f(0,4) = 0$

The equation of the graph $z = 8-4x-2y$

plane — $\boxed{4x+2y+z-8=0}$

(Recall: $ax+by+cz+d=0$ – equation of a plane)

ufficient to find 3 pts on this plane:

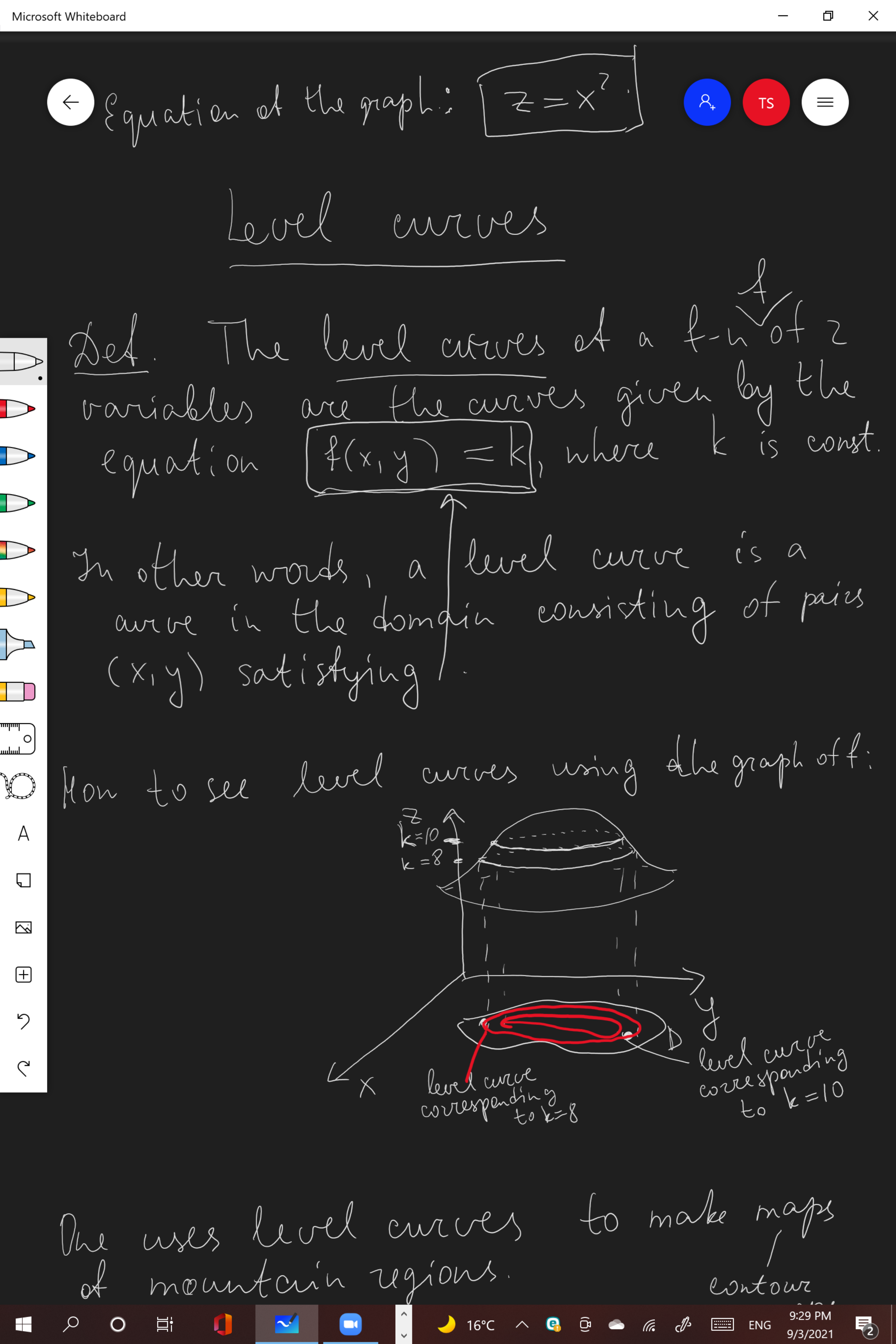
$(0,4,0)$. $x=0,\ y=4,\ z=0$.

$=0,\ y=0 \Rightarrow z=8$ $(0,0,8)$

$y=0,\ z=0 \Rightarrow x=2$ $(2,0,0)$

☐

oll: which of those surfaces is the graph $f(x,y) = x^2$?



Equation of the graph: $\boxed{z=x^2}$

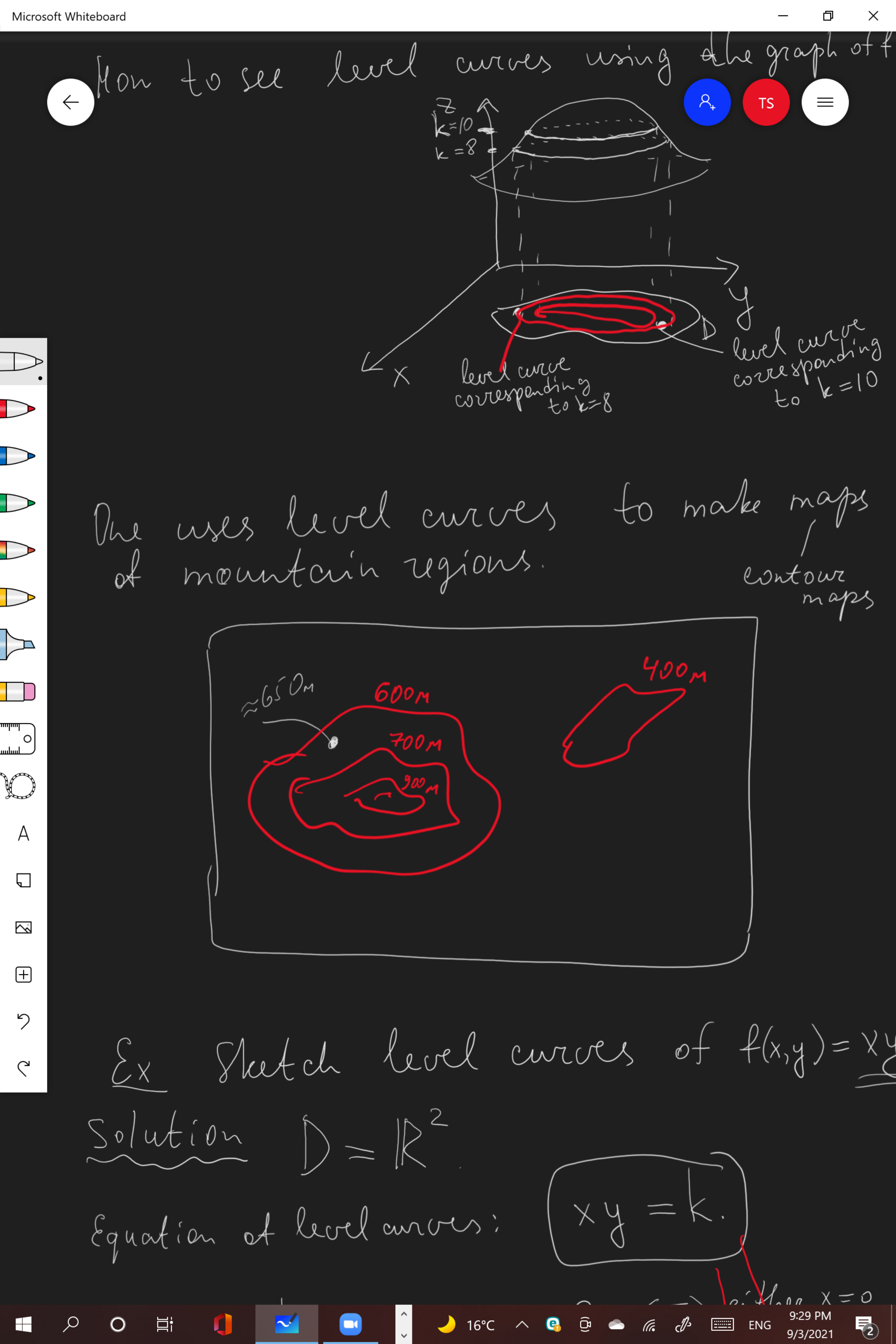
Level curves

Def. The level curves of a f-n f of 2 variables are the curves given by the equation $\boxed{f(x,y)=k}$, where k is const.

In other words, a level curve is a curve in the domain consisting of pairs (x, y) satisfying .

How to see level curves using the graph of f:

One uses level curves to make maps / contour of mountain regions.



How to see level curves using the graph of f

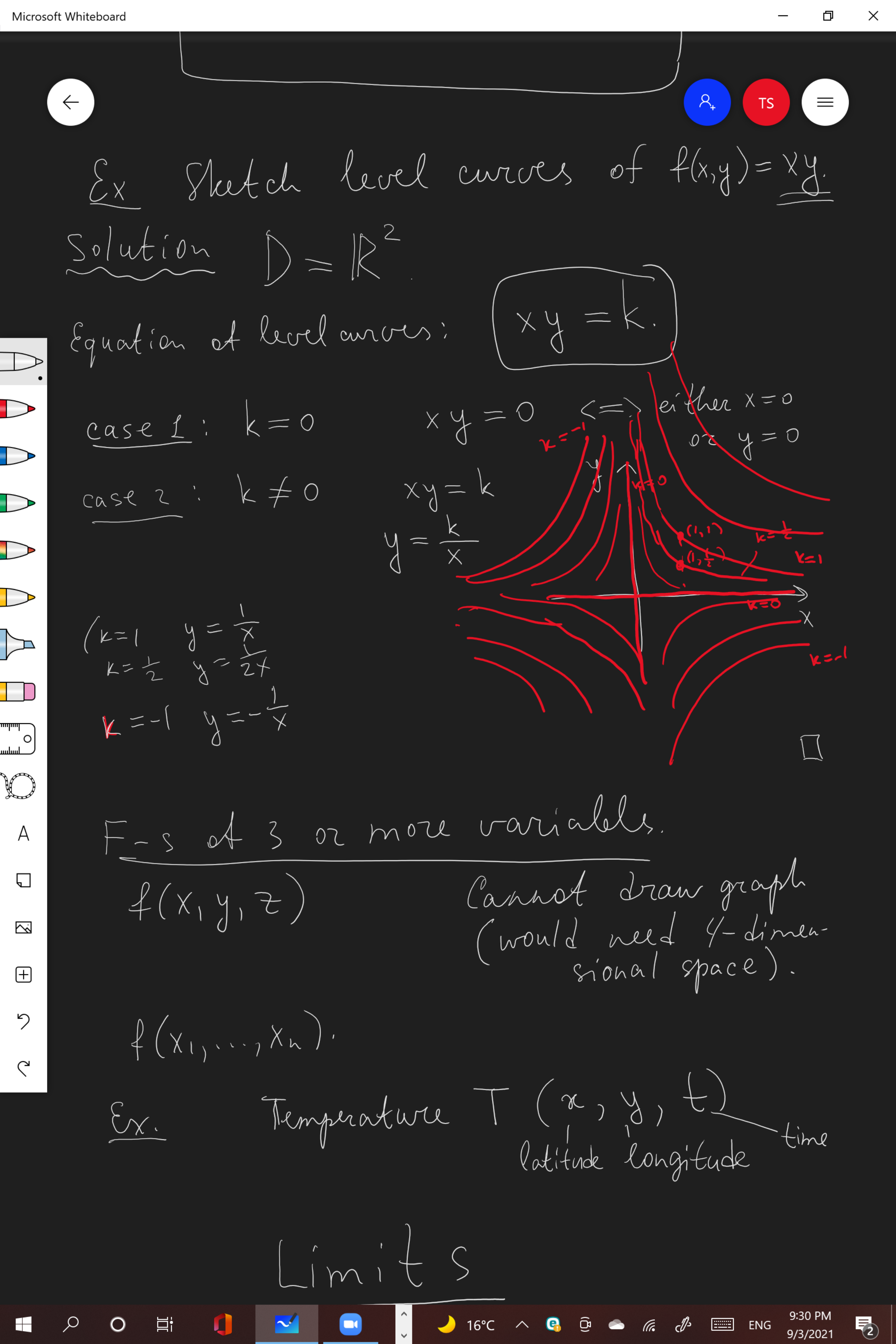
One uses level curves to make maps / contour maps of mountain regions.

Ex Sketch level curves of $f(x,y)=xy$

Solution $D=\mathbb{R}^2$.

Equation of level curves: $\boxed{xy=k.}$

either $x=0$



Ex Sketch level curves of $f(x,y) = xy$.

Solution $D = \mathbb{R}^2$.

Equation of level curves: $\boxed{xy = k.}$

case 1 : $k = 0$ $\quad xy = 0 \iff$ either $x = 0$ or $y = 0$

case 2 : $k \neq 0$ $\quad xy = k$

$$y = \frac{k}{x}$$

$(k = 1 \quad y = \frac{1}{x}$

$k = \frac{1}{2} \quad y = \frac{1}{2x}$

$k = -1 \quad y = -\frac{1}{x}$

□

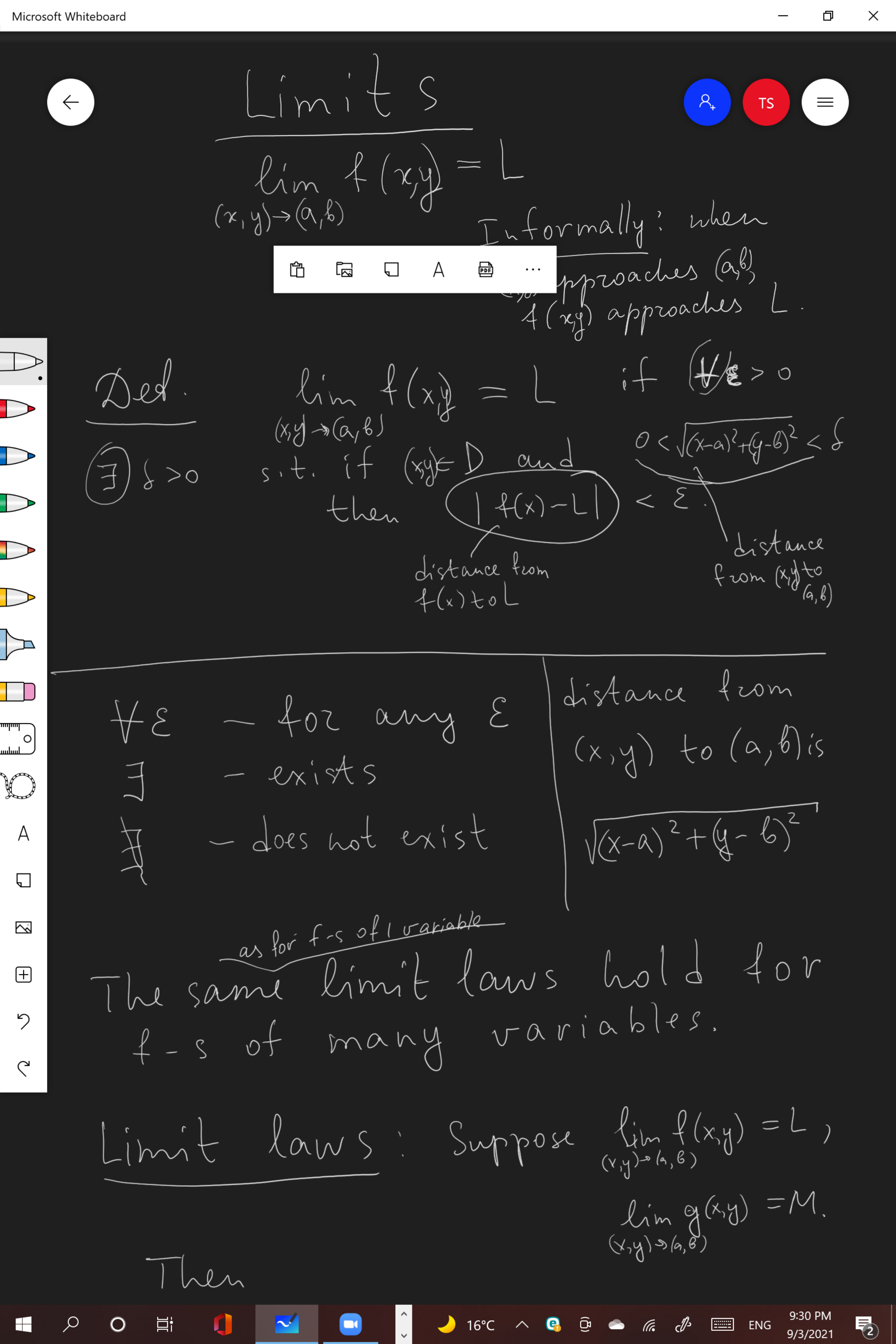
Fns of 3 or more variables.

$f(x, y, z)$ Cannot draw graph (would need 4-dimensional space).

$f(x_1, \dots, x_n)$.

Ex. Temperature $T(x, y, t)$ latitude longitude time

Limits



Limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Informally: when

pproaches (a,b), f(x,y) approaches L.

Def. $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $\forall \varepsilon > 0$

$\exists \delta > 0$ s.t. if $(x,y)\in D$ and $0 < \underbrace{\sqrt{(x-a)^2+(y-b)^2}}_{\text{distance from }(x,y)\text{ to }(a,b)} < \delta$

then $\underbrace{|f(x)-L|}_{\text{distance from } f(x) \text{ to } L} < \varepsilon$.

$\forall \varepsilon$ – for any ε

$\exists$ – exists

$\nexists$ – does not exist

distance from (x,y) to (a,b) is

$$\sqrt{(x-a)^2+(y-b)^2}$$

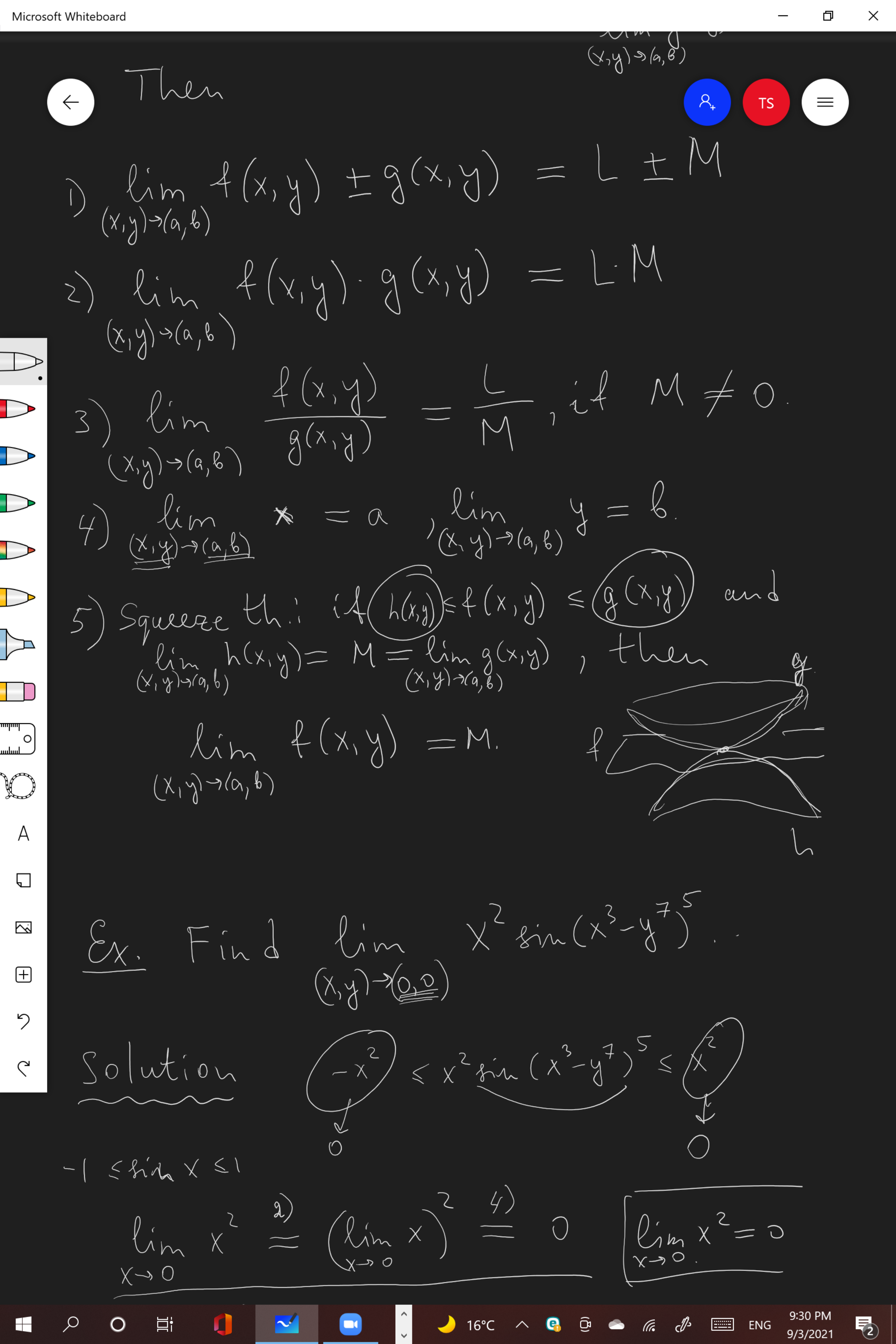
as for f-s of 1 variable

The same limit laws hold for f-s of many variables.

Limit laws: Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$,

$\lim_{(x,y)\to(a,b)} g(x,y) = M$.

Then



$(x,y)\to(a,b)$

Then

1) $\lim_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = L \pm M$

2) $\lim_{(x,y)\to(a,b)} f(x,y)\cdot g(x,y) = L\cdot M$

3) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, if $M \neq 0$.

4) $\lim_{(x,y)\to(a,b)} x = a$, $\lim_{(x,y)\to(a,b)} y = b$.

5) Squeeze th: if $h(x,y) \le f(x,y) \le g(x,y)$ and $\lim_{(x,y)\to(a,b)} h(x,y) = M = \lim_{(x,y)\to(a,b)} g(x,y)$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = M.$$

Ex. Find $\lim_{(x,y)\to(0,0)} x^2 \sin(x^3-y^7)^5$.

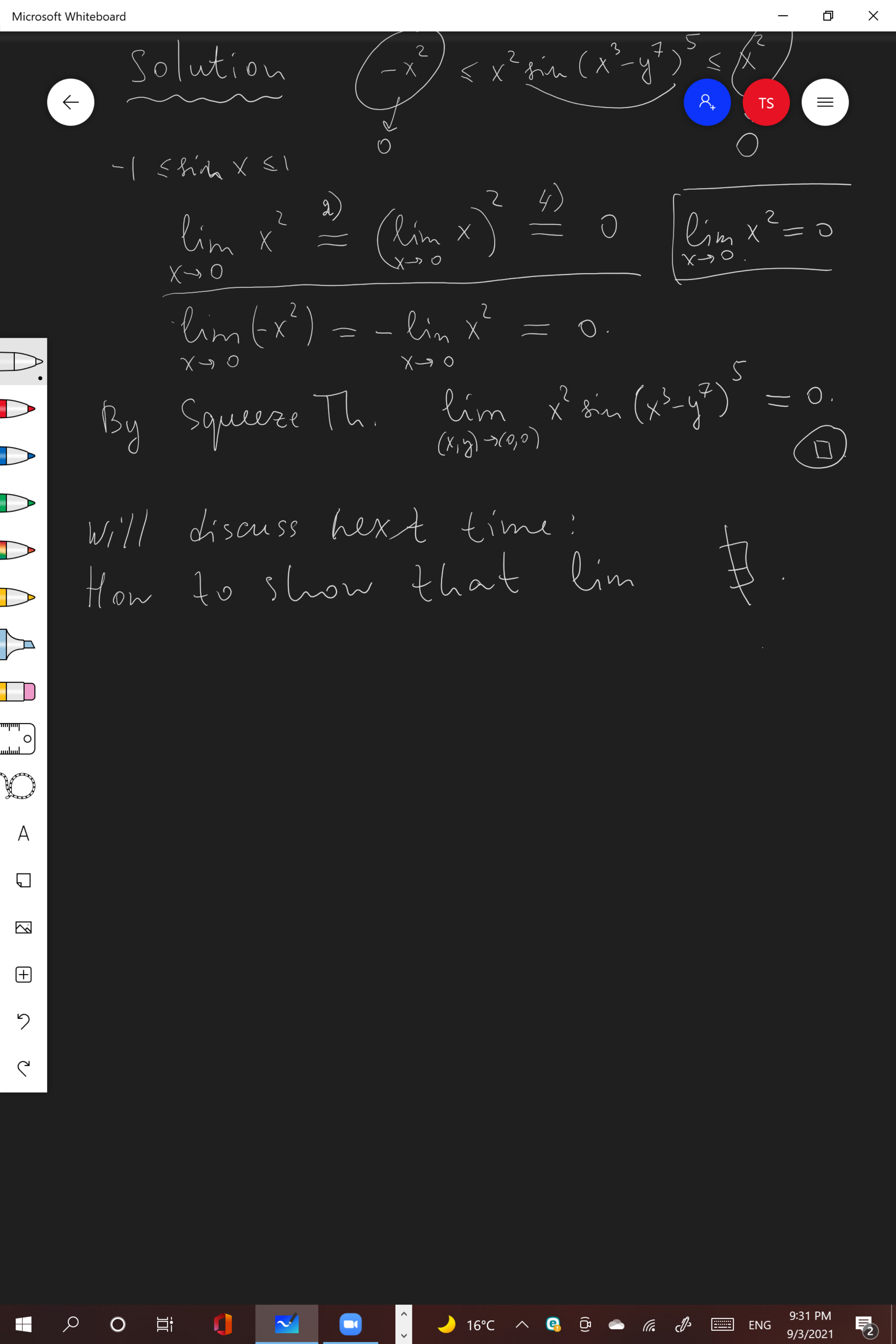
Solution

$$\underset{\downarrow\, 0}{-x^2} \le x^2 \sin(x^3-y^7)^5 \le \underset{\downarrow\, 0}{x^2}$$

$-1 \le \sin x \le 1$

$$\lim_{x\to 0} x^2 \overset{2)}{=} \left(\lim_{x\to 0} x\right)^2 \overset{4)}{=} 0$$

$$\boxed{\lim_{x\to 0} x^2 = 0}$$



Solution

$$\underset{\downarrow \atop 0}{-x^2} \le x^2 \underbrace{\sin(x^3-y^7)^5} \le \underset{\downarrow \atop 0}{x^2}$$

$-1 \le \sin x \le 1$

$$\lim_{x\to 0} x^2 \overset{2)}{=} \left(\lim_{x\to 0} x\right)^2 \overset{4)}{=} 0 \qquad \boxed{\lim_{x\to 0} x^2 = 0}$$

$$\lim_{x\to 0}(-x^2) = -\lim_{x\to 0} x^2 = 0.$$

By Squeeze Th. $\lim_{(x,y)\to(0,0)} x^2 \sin(x^3-y^7)^5 = 0.$ □

Will discuss next time:

How to show that lim ∄.