1^{st} lecture:

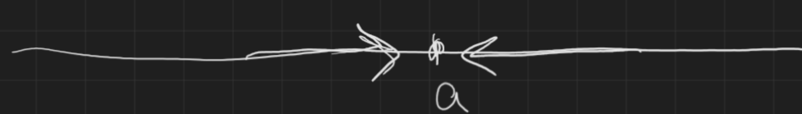
- $f(x, y)$, graphs, level curves.
- limits. Intuitively:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff$$

as (x,y) approaches (a,b), $f(x,y)$ approaches L.

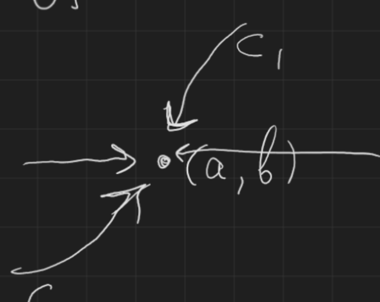
How to show that limit $\nexists$

Recall: $f(x)$, $x \to a$.



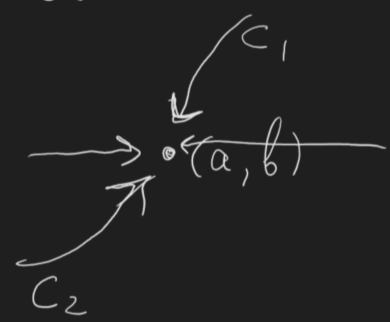
$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x), \text{ then } \lim_{x\to a} \nexists.$$

Similar idea for f-s of 2 variables.



$x \to a$

Similar idea for f-s of 2 variables.



$$\boxed{\begin{array}{l}\text{If } f(x,y) \to L_1 \text{ as } (x,y)\to(a,b) \text{ along a path } C_1 \text{ and } f(x,y)\to L_2 \text{ as } (x,y)\to(a,b) \text{ along a path } C_2 \text{ and } L_1 \neq L_2, \text{ then} \\ \lim_{(x,y)\to(a,b)} f(x,y) \ \nexists.\end{array}}$$

Ex. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \ \nexists$.

Solution $D = \{(x,y) \mid x^2+y^2 \neq 0\} = \{(x,y) \mid (x,y) \neq (0,0)\}$

Along the path $y=0$: $\frac{x^2-y^2}{x^2+y^2} = 1 \to 1$

Along the path $x=0$: $\frac{x^2-y^2}{x^2+y^2} = \frac{-y^2}{y^2} = -1 \to -1$

$(x,y)\to(0,0)$

Ex. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ $\nexists$.

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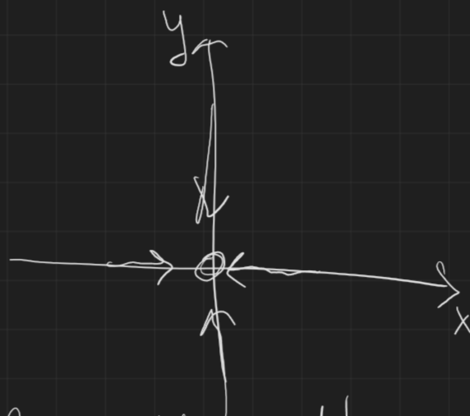

Along the path $\underline{y=0}$: $\dfrac{x^2-y^2}{x^2+y^2} = 1 \longrightarrow 1$

Along the path $x=0$: $\dfrac{x^2-y^2}{x^2+y^2} = \dfrac{-y^2}{y^2} = -1 \longrightarrow -1$

Hence $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ $\nexists$. $\square$

Ex. Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ $\exists$?

Solution $D=\{(x,y) \mid x^2+y^2\neq 0\} = \{(x,y) \mid (x,y)\neq(0,0)\}$

$\lim\limits_{(x,y)\to(0,0)} x^2+y^2 = 0$, $\lim\limits_{(x,y)\to(0,0)} xy = 0$

" $\dfrac{0}{0}$ "

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" $\frac{0}{0}$ "

Will try to show that the limit $\nexists$.

Along the path $y=0$: $\dfrac{xy}{x^2+y^2} = 0 \to 0$

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Poll: can we conclude that the limit $= 0$?

a) yes

b) no

Along the path $y=x$: $\dfrac{xy}{x^2+y^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2} \to \dfrac{1}{2}$

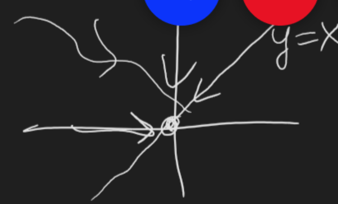
Hence the limit $\nexists$. $\square$

xy^2

the limit $= 0$?

a) yes

b) no

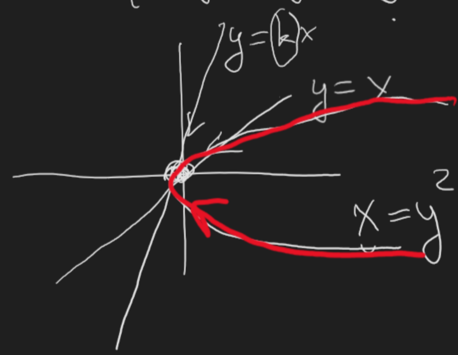


Along the path $y=x$: $\dfrac{xy}{x^2+y^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2} \to \dfrac{1}{2}$

Hence ... $\nexists$.

Ex. Show that $\lim\limits_{(x,y)\to(0,0)} \boxed{\dfrac{xy^2}{x^2+3y^4}}$ $\nexists$.

Solution $D = \{(x,y) \mid x^2+3y^4 \neq 0\} = \{(x,y) \mid (x,y) \neq (0,0)\}$



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Along the path $y=x$: $\dfrac{xy^2}{x^2+3y^4} = \dfrac{x^3}{x^2+3x^4} =$

$$= \frac{x}{1+3x^2} \longrightarrow 0$$

Along the path $y=kx$: $\dfrac{xy^2}{x^2+3y^4} = \dfrac{k^2x^3}{x^2+3k^4x^4} =$

$k^2 x$

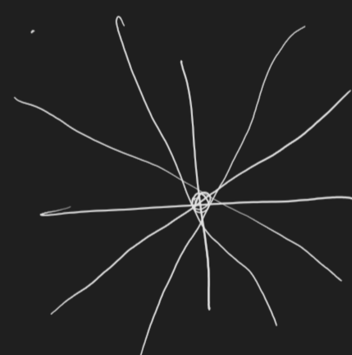
Along the path $y=kx$: $\dfrac{xy^2}{x^2+3y^4} = \dfrac{k^2x^3}{x^2+3k^4x^4}$

$$= \frac{k^2 x}{1+3k^4x^2} \to 0.$$

Along the parabola $x=y^2$: $\dfrac{xy^2}{x^2+3y^4} =$

$$= \frac{y^4}{y^4+3y^4} = \frac{1}{4} \to \frac{1}{4}.$$

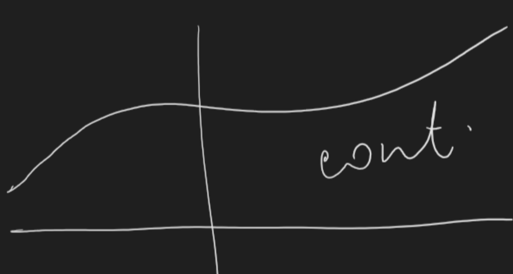
Hence the limit $\nexists$.

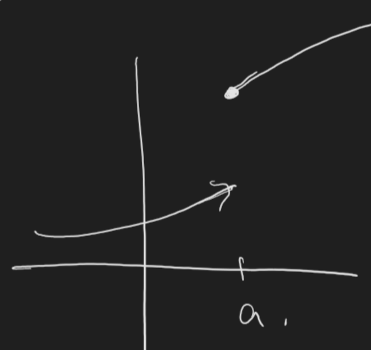


Continuity for f-s of many variables

Recall: $f(x)$ is continuous at $x=a$

if $\lim\limits_{x\to a} f(x) = f(a)$.





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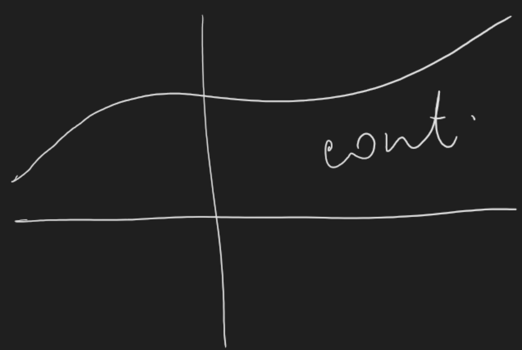

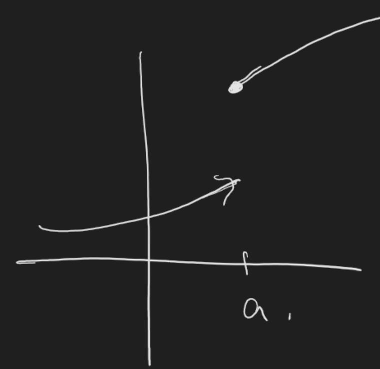

Def. Let f be a f-n of 2 variables.

f is continuous at point (a,b) in the domain of f if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

We say f is continuous in D if it is continuous at each point in D.

We say f is continuous if it is continuous in $\mathbb{R}^2$.

Using limit laws we see that

Using limit laws we see that if f, g, h are continuous in their domains, then

1) $f \pm g$ is contin.

(Indeed, $\lim\limits_{(x,y)\to(a,b)} f(x,y)+g(x,y) \overset{\text{limit law 1)}}{=}$

$= \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y) = f(a,b) + g$

$= (f+g)(a,$

2) $f \cdot g$ is contin.

3) $h(f(x,y))$ is contin.

4) $\frac{f}{g}$ is contin.

in their domains.

Examples of contin. f-s: 1) $\lim\limits_{(x,y)\to(a,b)} x = a$,

$\lim\limits_{(x,y)\to(a,b)} y = b$

It means x and y are contin. f-s.

2) $x^2y^7 - 3xy^3$ is contin.

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$\lim\limits_{(x,y)\to(a,b)} y = b$

It means x and y are contin. f-s.

2) $x^2y^7 - 3xy^3$ is contin.
More generally, any polynomial in several variables is contin.

3) $\dfrac{x}{1-y}$ is contin. on $D = \{(x,y) \mid y \neq 1\}$.

4) $\dfrac{x}{1+y^2}$ is contin. on $\mathbb{R}^2$.

5) $e^{\frac{x}{1+y^2}}$ is contin. on $\mathbb{R}^2$.

Ex. Find $\lim\limits_{(x,y)\to(0,0)} x^2 \sin(x^3-y^7)^5$

(Already found before using Squeeze Th. Now can shorter using continuity).

Solution $f(x,y) := x^2 \sin(x^3-y^7)^5$ is contin.

Hence $\lim\limits_{(x,y)\to(0,0)} x^2 \sin(x^3-y^7)^5 = f(0,0) = 0.$ $\square$

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Ex. Determine whether the following f-s are contin. at $(0,0)$:

a) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+3y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

b) $f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^2+y^2}, & (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0) \end{cases}$

c) $f(x,y) = \begin{cases} =//= \\ 0, & (x,y)=(0,0) \end{cases}$?

Solution Need to check $\boxed{\lim\limits_{(x,y)\to(0,0)} f(x,y)} \stackrel{?}{=}$

a) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+3y^4}$ $\nexists$, as shown before.

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Solution Need to check $\boxed{\lim_{(x,y)\to(0,0)} f(x,y)} \overset{?}{=} f$

a) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4} \nexists$,

as shown before.

Hence f is not contin. at $(0,0)$

b) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} =$

$= \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = \lim_{(x,y)\to(0,0)} x^2-y^2 = 0$

$f(0,0) = 1 \neq 0.$

Hence f is not contin. at $(0,0)$.

c) $\lim_{(x,y)\to(0,0)} f(x,y) \overset{b)}{=} 0$

$\|$

$f(0,0) = 0$

Hence f is contin. at $(0,0)$.

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Partial derivatives.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x,y)$

Def. Let f be a f-n of 2 variables. The partial derivative of f with respect to x, denoted f_x, is a f-n of 2 variables given by the formula

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

The partial derivative of f w.r.t. y, denoted f_y, is a f-n of 2 variables given by the formula $f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$.

We see, that In the definition of f_x y is treated as a constant and x as a variable. Similar for f_y.

Equivalent def. Let f be a f-n of 2 variables. Fix y and let $h(x) := f(x,y)$. The partial

Equivalent def. Let f be a f-n of 2 variables. Fix y an[d] let $h(x) := f(x,y)$. The partial derivative w.r.t. x is

$$f_x = h'.$$

Similar for f_y.

To find f_x derive f assuming y is a constant.

To find f_y derive f assuming x is a constant.

Ex. Find partial derivatives.

a) $f(x,y) = x^2 y + y^3$.

$$f_x(x,y) = 2xy$$

$$f_y(x,y) = x^2 + 3y^2.$$

b) $f(x,y) = \sin\left(\underline{x^2 y - 1}\right)$

$$f_x(x,y) \overset{\text{chain rule}}{=} \cos(x^2 y - 1) \cdot 2xy$$

$$f_y(x,y) = \cos(x^2 y - 1) \cdot x^2$$

$f(x,y) =$

b) $\sin(x^2 y - 1)$

$$f_x(x,y) \overset{\text{chain rule}}{=} \cos(x^2 y - 1) \cdot 2xy$$

$$f_y(x,y) = \cos(x^2 y - 1) \cdot x^2$$

c) $f(x,y,z) = x^2 z + x \cos y - y z^5$

$$f_x = 2xz + \cos y$$

$$f_y = -x \sin y - z^5$$

$$f_z = x^2 - y \cdot 5z^4$$

Poll $\quad f(x,y) = e^{3xy}$. What is f_x?

a) e^{3xy}

b) $3y\, e^{3xy}$

c) $3x\, e^{3xy}$

$$f_x = e^{3xy} \cdot 3y$$

Another notation:

$$f_x = e^{3xy} \cdot 3y$$

Another notation:

$$\boxed{f_x = \frac{\partial f}{\partial x}} = f'_x$$

$$\boxed{f_y = \frac{\partial f}{\partial y}} = f'_y$$