Higher partial derivatives

Recall: $f(x)$ $\qquad f'' = (f')'$

For $f(x,y)$ we can consider partial derivatives of partial derivatives:

$$(f_x)_x, \quad (f_x)_y, \quad (f_y)_x, \quad (f_y)_y$$

— 4 2nd order partial derivatives.

Notation: $f_x = \frac{\partial f}{\partial x}, \; f_y = \frac{\partial f}{\partial y}$

$$(f_x)_x = \boxed{f_{xx}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \boxed{\frac{\partial^2 f}{\partial x^2}}$$

$$(f_x)_y = \boxed{f_{xy}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \boxed{\frac{\partial^2 f}{\partial y \partial x}}$$

$$(f_y)_x = \boxed{f_{yx}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \boxed{\frac{\partial^2 f}{\partial x \partial y}}$$

$$(f_y)_y = \boxed{f_{yy}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \boxed{\frac{\partial^2 f}{\partial y^2}}$$

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Ex. Find 2^{nd} partial derivatives of the f-n $f(x,y) = xe^y$

Solution At first we find partial derivatives: $f_x = e^y$

$f_y = xe^y$

Now 2^{nd} partial derivatives:

$$f_{xx} = (f_x)_x = (e^y)_x = 0$$

$$f_{xy} = (f_x)_y = (e^y)_y = e^y$$

$$f_{yx} = (f_y)_x = (xe^y)_x = e^y$$

$$f_{yy} = (f_y)_y = (xe^y)_y = xe^y. \quad \square$$

Clairot's Theorem (Schwarz Th.)

If (f_{xy} and f_{yx} are continuous near (a,b)), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Vague explanation:

suppose $f(x,y)$ is a polynomial,

Say $f(x,y) = x^n y^m$

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Clairot's Theorem (Schwarz Th.)

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Vague explanation:

suppose $f(x,y)$ is a polynomial,

say $f(x,y) = x^n y^m$.

Let us compute f_{xy} and f_{yx}.

$$f_x = n x^{n-1} y^m, \quad f_y = m x^n y^{m-1}$$

$$f_{xy} = (f_x)_y = (n x^{n-1} y^m)_y = nm x^{n-1} y^{m-1}$$

$$f_{yx} = (f_y)_x = (m x^n y^{m-1})_x = mn x^{n-1} y^{m-1}$$

Thus Clairaut's Th. obviously holds for all polynomials.

Since any contin. f-n can be approximated by polynomials, it doesn't sound surprising that Clairaut's Th. holds for other f-s.

(It is not a proof!)

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How to use Clairaut's Th.: we can compute only one of f_{xy} and f_{yx}, the other one is the same.

In most examples we can see immediately that f_{xy} and f_{yx} must be continuous, without computing them. Namely if $f(x,y)$ is a combination of exp, sin, cos, ..., then all its partial derivatives of any order are continuous.

Ex. Find f_{xy} and f_{yx} for $f(x,y) = y \sin(x+y)$

Solution $f_x = y \cos(x+y)$

$f_y = \sin(x+y) + y\cos(x+y)$.

$$f_{xy} = (f_x)_y = (y\cos(x+y))_y = \boxed{\cos(x+y) - y\sin(x+y)}$$

By Clairaut's Th. $f_{yx} = f_{xy}$. $\square$

Partial derivatives of order

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Partial derivatives of order 3, 4, 5, ...

Of order 3: for example

$(f_{xx})_x = f_{xxx}, \quad f_{xyx}, \quad f_{yxy}, \ldots$

Question: How many partial derivatives of order 3 $f(x,y)$ has? 8

order 5 $\quad 2^5 = 32$

$f_{000\ldots}$

Some of mixed partial derivatives of higher order coinside, for example

$f_{xyx} \stackrel{?}{=} f_{xxy} \quad = f_{yxx}.$

$f_{xyx} = (f_{xy})_x \underset{\text{Clairaut's Th.}}{=} (f_{yx})_x =$

$= f_{yxx}.$

$g^{10}.$

Geometrical sense of partial derivatives.

Recall: geometrical sense of $f'(a)$, for a f-n of 1 variable.

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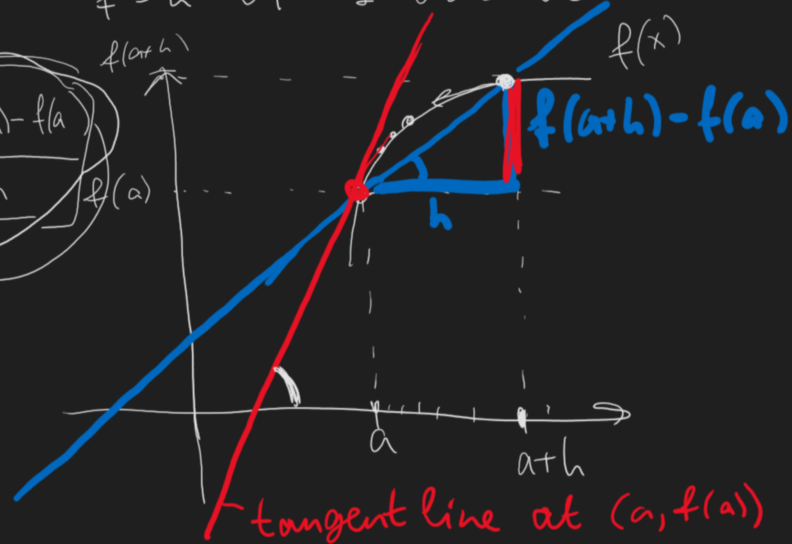
Geometrical sense of partial derivatives.

Recall: geometrical sense of $f'(a)$, for a f-n of 1 variable.

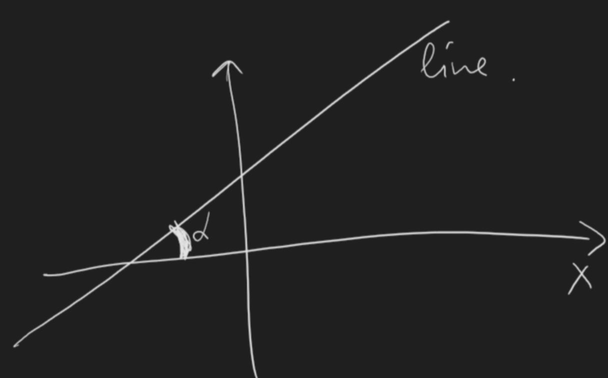
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

the slope of the secant line

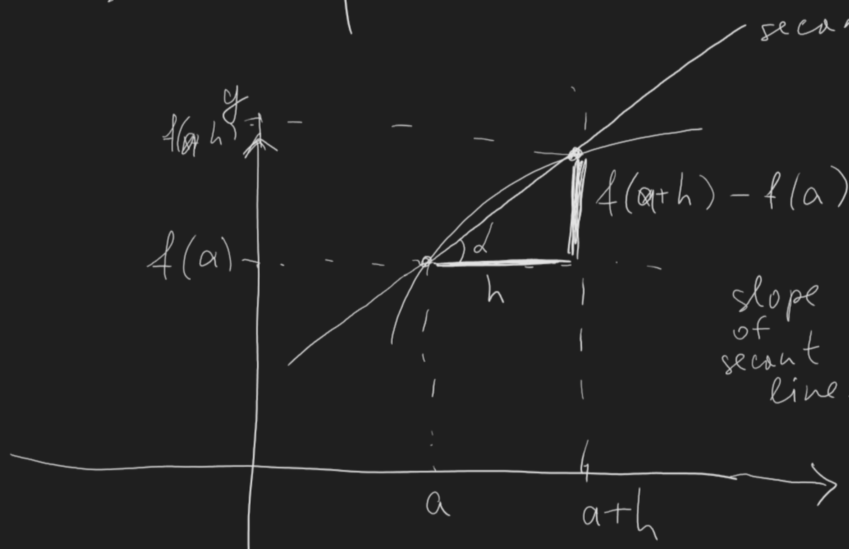


$$\boxed{f'(a)} = \lim_{h\to 0} \text{ of the slopes of secant lines}$$

$= \boxed{\text{the slope of the tangent line at } (a, f(a))}$

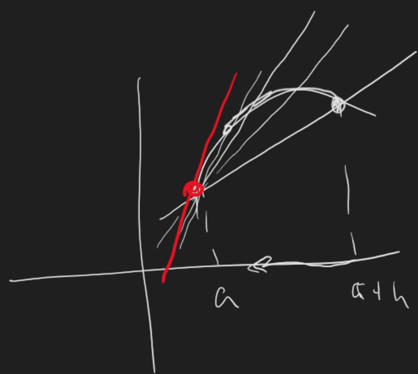


$\boxed{tg\,\alpha}$ – slope of the line



slope of secant line. $= \dfrac{f(a+h)-f(a)}{h}$.

$$f'(a) = \lim_{h\to 0} \boxed{\frac{f(a+h)-f(a)}{h}} = \lim_{h\to 0} \text{ of slopes of secant lines}$$



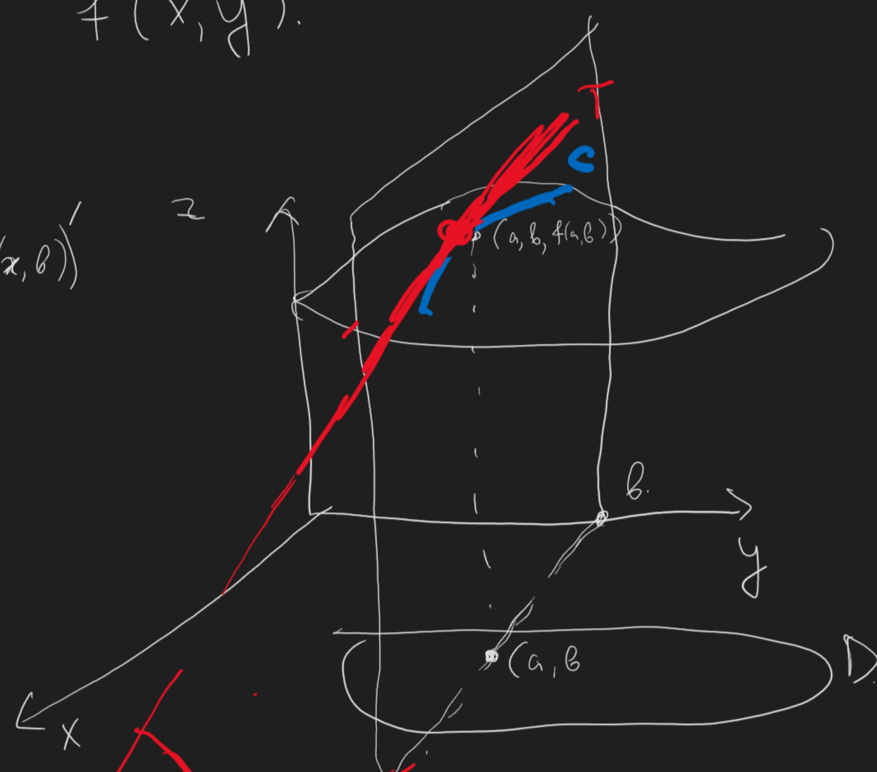
slopes of secant lines $\xrightarrow[h\to 0]{}$ slope of tangent line

$f'(a)$ = the slope of the tangent line.

Now $f(x,y)$.

$f_x(x,b) = (f(x,b))'$

Fix $y=b$.
plane

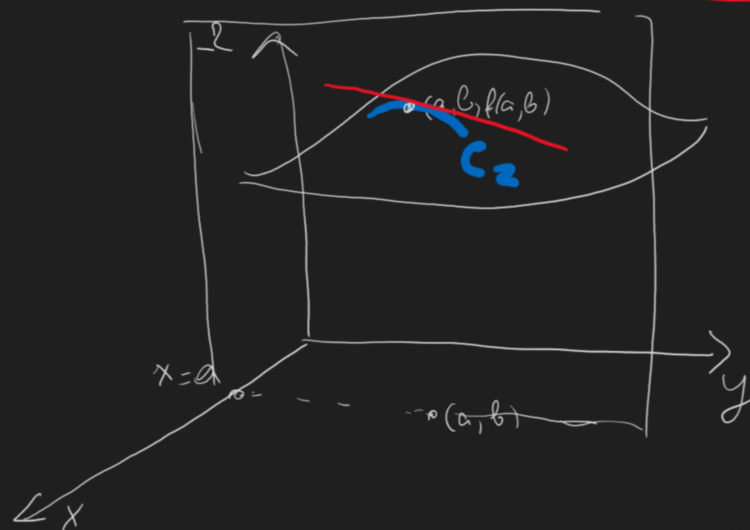


C - intersection of the graph with the plane $y=b$.

$f_x(a,b)$ is the slope of the tangent line to the curve C at $(a,b,f(a,b))$, where C is the intersection of the graph of f with the plane $y=b$

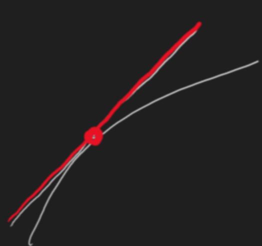
$f_x(a,b)$ is the slope of the tangent to the curve C at $(a,b,f(a,b))$, where C is the intersection of the graph of f with the plane $y=b$

$f_y(a,b)$



$f_y(a,b)$ is the slope of the tangent line to the curve C_2 at $(a,b,f(a,b))$, where C_2 is the intersection of the graph of f with the plane $x=a$.

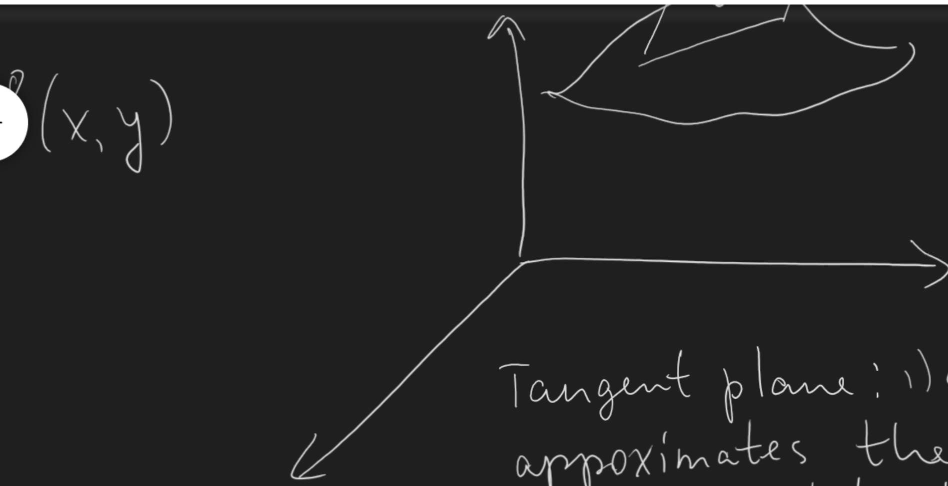
Tangent planes



tangent line: 1) limit of secant lines

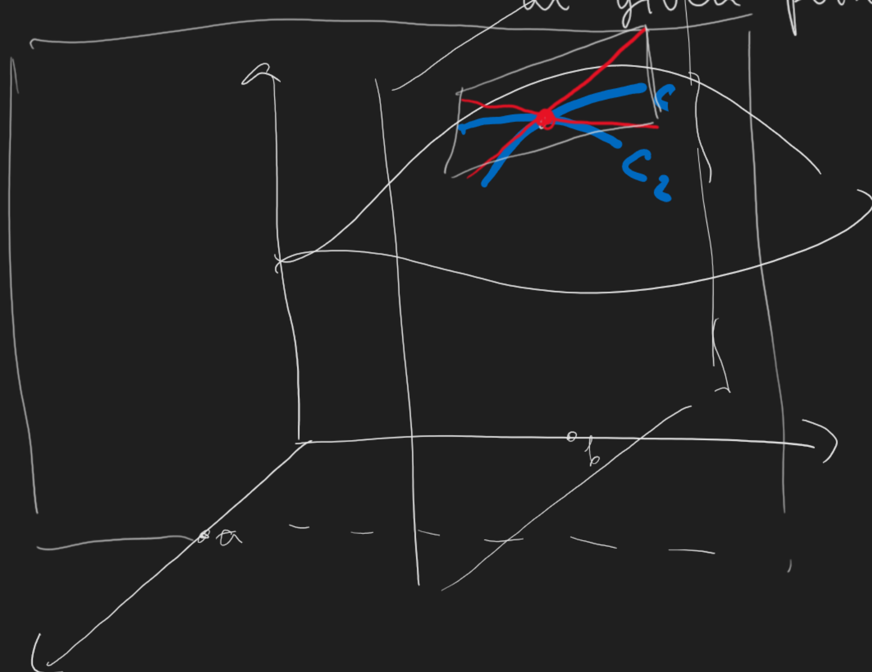
2) a line that approximates the graph of f at $(a, f(a))$ the best.

3) (visually) a line that touches the graph



Tangent plane: 1) a plane that appoximates the graph of f at given point the best

2) (visually) a plane that touches the graph at given point.



Tangent plane must contain 2 tangent lines (to the curves c and c_2).

Tangent planes to more general surfaces (not only graphs of f-s) will be discussed later.

A graph of a f-n might not have tangent

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A graph of a f-n might not have tangent plane.

Solution.

Ex. a) Show that f_x and f_y exist.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y)=(0,0). \end{cases}$$

b) Show that f_x and f_y are not contin. at $(0,0)$.

a) At any point $\neq (0,0)$ f_x and f_y $\exists$. (and easy to find)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \underline{\underline{0}}$$

Similarly $f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = 0.$

$(x,y) \neq (0,0)$

$$f_x = \left(\frac{xy}{x^2+y^2}\right)_x = \frac{y(x^2+y^2)-2x^2y}{(x^2+y^2)^2} = \frac{y^3-x^2y}{(x^2+y^2)^2} = \boxed{\frac{y(y^2-x^2)}{(x^2+y^2)^2}}.$$

$f_y = \ldots$

b) Show that f_x is not contin.

need only to check continuity at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f_x(x,y) \stackrel{?}{=} f_x(0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

Along $x=0$: $\dfrac{y^3}{y^4} = \dfrac{1}{y} \to \infty$, as $y\to 0$

So lim along this path $\nexists$.

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

Along $x=0$: $\frac{y}{y^4} = \frac{1}{y^3} \to \infty$, as $y \to 0$

So lim along this path $\nexists$.

Hence $\lim_{(x,y)\to(0,0)} f_x(x,y) \nexists$.

In fact, if f_x and f_y are contin.,
then there is a tangent plane.

Previous lecture:

- How to show that $\lim \nexists$
- Continuity
- Partial derivatives

Def. $f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$.

$f_y = \cdots$

Equiv. def. Fix y and consider

$h(x) = f(x,y)$

Then $f_x = h'$.

Simply speaking: To find f_x derive f assuming y is const.

To find f_y ....

Higher partial derivatives

Recall: $f(x)$ $\qquad f'' = (f')'$

For $f(x,y)$ we can consider partial derivatives of partial