$\boxed{14.3.71}$

Compute the following partial derivative

$$f(x,y,z) = x y^2 z^3 + \arcsin(x\sqrt{z})$$

$$f_{xzy} \ ?$$

1) Domain : - $\sqrt{\ }$ is defined on $\mathbb{R}^+$

- arcsin is defined on $[-1, 1]$

So f is defined on $D = \{(x,y,z) \in \mathbb{R}^3, z \geq 0, x\sqrt{z} \in [-1,1]\}$

2) there is no y-variable in arcsin

So we would like to use Clairaut's theorem

<u>Recall</u> Clairaut's Theorem

Let f be a function defined on D and $(a,b) \in D$

if f_{xy} and f_{yx} are continuous then $f_{xy}(a,b) = f_{yx}(a,b)$

3) Since there is no y in arcsin part, any higher order partial derivatives will erase the part with arcsin

4) Polynomials have partial derivatives of any order which are continuous

So f_{xyz}, f_{yxz}, ... are continuous

So we can apply Clairaut's theorem

and $f_{xyz} = f_{yxz}$

⚠ $\arcsin'(t) = \frac{1}{\sqrt{1-t^2}}$ on $]-1,1[$

$$f_{yxz}(x,y,z) = (f_y)_{xz}(x,y,z)$$

$$f_y(x,y,z) = \frac{\partial x y^2 z^3}{\partial y} + \overbrace{\frac{\partial \arcsin(x\sqrt{z})}{\partial y}}^{0}$$

(because of linearity)

$$= 2xyz^3$$

$$f_{yx}(x,y,z) = 2yz^3$$

$$f_{yxz}(x,y,z) = 6yz^2 \overset{\text{Clairaut theorem}}{=\!=} f_{xzy}(x,y,z)$$

$\boxed{14.3.75}$

Show that $u(t,x) = e^{-\alpha^2 k^2 t} \sin(kx)$ satisfies the heat equation $u_t = \alpha^2 u_{xx}$

$$u_t(t,x) = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin(kx)$$

$$u_x(t,x) = k\, e^{-\alpha^2 k^2 t} \cos(kx)$$

$$u_{xx}(t,x) = -k^2 e^{-\alpha^2 k^2 t} \sin(kx)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial e^{-\alpha^2 k^2 t} \sin(kx)}{\partial x} = e^{-\alpha^2 k^2 t} \frac{\partial \sin(kx)}{\partial x}$$

so $u_t = \alpha^2 u_{xx}$ and u satisfies heat equation

chain rule $f(g(x))' = g'(x) f'(g(x))$

here $g(z) = kz$ and $f(z) = \sin(z)$

$\boxed{14.3.103}$

We have $f(x,y) = x(x^2+y^2)^{-3/2} e^{\sin(x^2 y)}$

Compute $f_x(1,0)$

<u>Recall</u> : 1) $f_x(a,b) = g'(a)$ where $g(x) = f(x,b)$

2) $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

$f_x(1,0) = g'(1)$ with $g(x) = f(x,0)$

$= x(x^2+0^2)^{-3/2} e^{\sin(x^2 \times 0)}$

$= x(x^2)^{-3/2} = x\, x^{-3} = x^{-2}$

$g'(x) = -2x^{-3}$

<u>recall</u> $h(x) = x^\alpha$ then $h'(x) = \alpha x^{\alpha-1}$ $\forall \alpha \in \mathbb{R}$

$g'(1) = -2$ so $f_x(1,0) = -2$

$\boxed{14.3.104}$

$f(x,y) = \sqrt[3]{x^3+y^3}$ compute $f_x(0,0)$

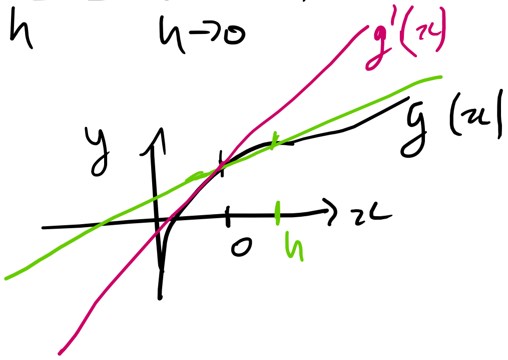
$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{(0+h)^3 + 0^3} - \sqrt{0^3+0^3}}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

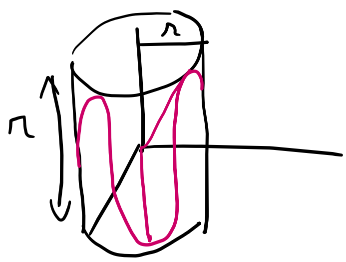
$$f_x(x,0) = g(x)$$



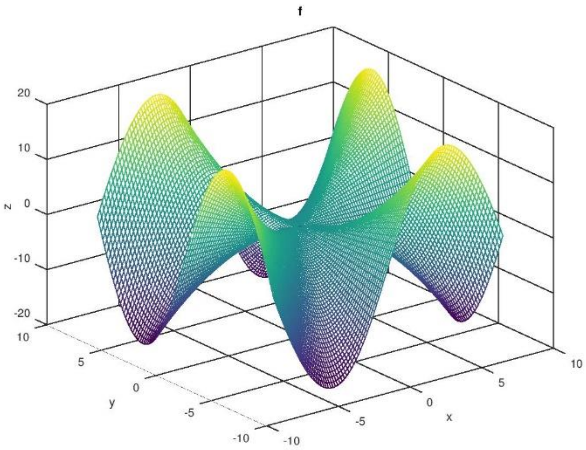
$$\boxed{14.3.105}$$

$$f(x,y) = \frac{x^3 y - x y^3}{x^2+y^2}$$

$$f(r\cos\theta, r\sin\theta) = \frac{r^2 \sin(4\theta)}{4}$$



a)



with Octave software

```
tx = ty = linspace (-8, 8, 101)';
[x, y] = meshgrid (tx, ty);
tz = ((x.^3 .* y ).-(x .* y.^3 ))./(x.^2 .+ y.^2);
figure(1);
mesh (tx, ty, tz);
xlabel ("x");
ylabel ("y");
zlabel ("z");
title ("f");
```

$\rightarrow$ defines the grid for (x, y)

$\rightarrow$ computes the value of f on every points of the grid

b) $f_x(x,y) = g'(x)$ with $g(x) = f(x,y)$ y fixed

<u>recal</u> $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

$$f_x(x,y) = \frac{(3x^2 y - y^3)(x^2+y^2) - 2x(x^3 y - xy^3)}{(x^2+y^2)^2}$$

$$= \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{(x^3 - 3xy^2)(x^2+y^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2}$$

c) $$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{\frac{h^3\times 0 - h\times 0^3}{h^2+0^2}}^{=0} - 0}{h}$$

$$= \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

$$= g'(0) \quad \text{with } g(y) = f(0,y)$$

$$= \frac{0^3y - 0y^3}{0^2+y^2} = 0$$

$$g'(y) = 0 \quad \text{so } g'(0) = 0$$

d) 1) Derive $(f_x)_y$

or 2) using equation 2 and 3

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h}$$

$(f_x)_y$

$$= \lim_{h\to 0} \frac{\left(-\frac{h^5}{h^4}\right)}{h} = \lim_{h\to 0} -1 = -1$$

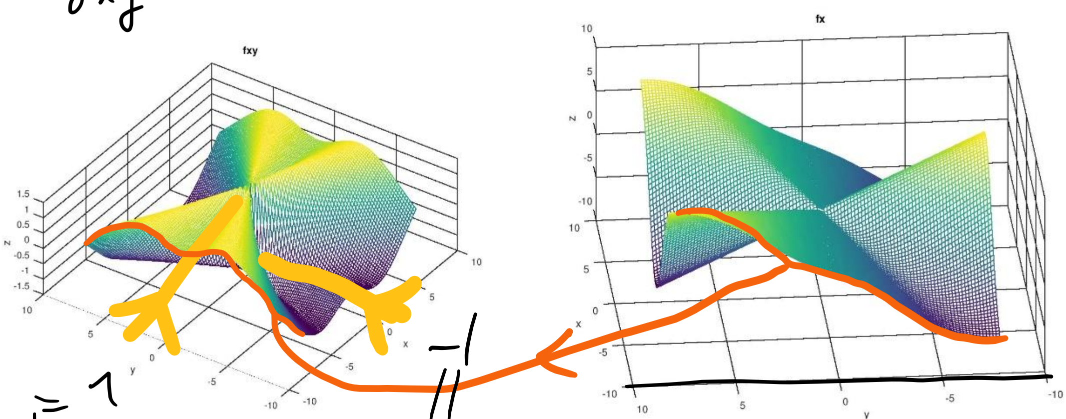
$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$ #

$$= \lim_{h\to 0} \frac{\frac{h^5}{h^4}}{h} = \lim_{h\to 0} 1 = 1$$

So $f_{xy}(0,0) \neq f_{yx}(0,0)$

We will see that f_{xy} and f_{yx} are not continuous so the Clairaut's theorem does not apply.

So f_{xy} is not continuous in $(0,0)$



limit on y-direction $= 1$ $\neq$ limit on x-direction -1

f_x graph

$$f_{xy}(x,y) = \frac{1}{(x^2+y^2)^4} \times \left[(x^4+12x^2y^2-5y^4)(x^2+y^2)^2 - 4y(x^2+y^2)(x^4y+4x^2y^3-y^5)\right]$$

$$f_{yx}(x,y) = \frac{1}{(x^2+y^2)^4} \times \left[(5x^4-12x^2y^2-y^4)(x^2+y^2)^2 - 4x(x^2+y^2)(x^5-4x^3y^2-xy^4)\right]$$

Code Octave for the figures

```
tx = ty = linspace (-8, 8, 101)';
[x, y] = meshgrid (tx, ty);
tz = ((x.^3 .* y ).-(x .* y.^3 ))./(x.^2 .+ y.^2);
figure(1);
mesh (tx, ty, tz);
xlabel ("x");
ylabel ("y");
zlabel ("z");
title ("f");

txyz = ((x.^4.+12.*x.^2.*y.^2 .- 5.*y.^4 ).*(x.^2 .+ y.^2).^2.-4.*y.*(x.^2 .+ y.^2 ).*(x.^4.*y .+
4.*x.^2.*y.^3.-y.^5))./(x.^2 .+ y.^2).^4;
figure(2);
mesh (tx, ty, txyz);
xlabel ("x");
ylabel ("y");
zlabel ("z");
title ("fxy");

tyxz = ((5.*x.^4.-12.*x.^2.*y.^2 .- y.^4 ).*(x.^2 .+ y.^2).^2.-4.*x.*(x.^2 .+ y.^2 ).*(x.^5 .- 4.*x.^3.*y.^2.-
x.*y.^4))./(x.^2 .+ y.^2).^4;
figure(3);
mesh (tx, ty, tyxz);
xlabel ("x");
ylabel ("y");
zlabel ("z");
title ("fyx");

txz = (x.^4.*y.+4.*x.^2.*y.^3 .- y.^5 )./(x.^2 .+ y.^2).^2;
figure(4);
mesh (tx, ty, txz);
xlabel ("x");
ylabel ("y");
zlabel ("z");
title ("fx");
```