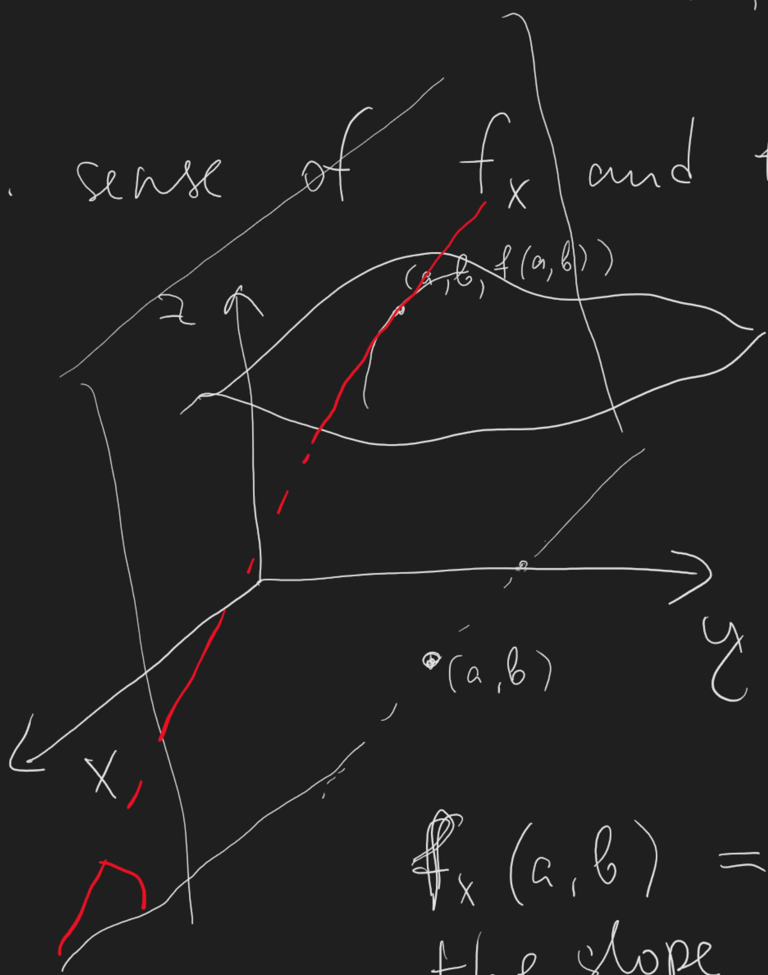
Previous lecture:

- higher partial derivatives. f_{xy}, f_{yx}, f_{xx}, f_{yy}.

- geometr. sense of f_x and f_y.



plane. $y=b$

$f_x(a,b) =$ the slope of this tangent line.

- a bit about tangent plane

• a bit about tangent plane

Differentiability

Recall: $y=f(x)$ differentiable $\iff$ has derivative.

$$y'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Δx - increment of x ($\Delta x \neq 0$).

The corresponding increment of y is

$$\Delta y = f(a+\Delta x) - f(a),$$

$$\boxed{y'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

$$\lim_{\Delta x \to 0} \underbrace{\frac{\Delta y}{\Delta x} - y'(a)}_{\varepsilon} = 0$$

f-n of Δx

Then $\varepsilon \to 0$ as $\Delta x \to 0$.

$\Delta x \to 0$

ε f-n of Δx

$$\varepsilon := \frac{\Delta y}{\Delta x} - y'(a).$$ Then $\varepsilon \to 0$ as $\Delta x \to 0$.

Multiple by Δx:

$$\varepsilon \Delta x = \Delta y - y'(a)\Delta x$$

$$\boxed{\Delta y = y'(a)\Delta x + \varepsilon \Delta x, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0}$$

Differentiability $\Longleftrightarrow$ Δy can be written in the form.

Now will formulate differentiability for f-s of 2 variables.

$z = f(x,y)$.

Δx – increment in x, Δy – increment in y.

Δz – the corresponding increment in z.

Namely

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b).$$

Def. $z = f(x,y)$ is differentiable

$\Delta z =$ … $y) - f(a, b)$.

Def. $z = f(x,y)$ is differentiable at (a,b) if its increment can be written in the form

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Remark: here ε_1 and ε_2 depend on Δx and Δy, so $\varepsilon_1 = \varepsilon_1(\Delta x, \Delta y)$ and $\varepsilon_2 = \varepsilon_2(\Delta x, \Delta y)$.

Compare with case of 1 variable:

$$\Delta y = f'(a)\,\Delta x + \varepsilon \Delta x$$

Ex. (considered at the end of previous lecture).

$$z = f(x,y) = \begin{cases} \boxed{\dfrac{xy}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(Already know: partial derivatives $\exists$
$f_x(0,0) = 0 = f_y(0,0)$.
f_x and f_y are not contin. near $(0,0)$.)

Microsoft Whiteboard

TS

Ex. (considered at the end of previous lecture).

$$z = f(x,y) = \begin{cases} \boxed{\dfrac{xy}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(Already know: partial derivatives $\exists$

$f_x(0,0) = 0 = f_y(0,0)$.

f_x and f_y are not contin. near $(0,0)$.)

Will show now: $z = f(x,y)$ is not differentiable at $(0,0)$.

$$\Delta z \overset{?}{=} \underbrace{f_x(0,0)}_{=0}\Delta x + \underbrace{f_y(0,0)}_{=0}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0) \quad \text{= def. of increment.}$$

$$= f(\Delta x, \Delta y) = \frac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2}$$

(assume $(\Delta x, \Delta y) \neq (0,0)$)

$$\boxed{\frac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2}} \overset{?}{=} \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

lim $\nexists$.

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

$\downarrow$

0

17°C ENG 11:41 PM 9/7/2021

$$\boxed{\frac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2}}$$

lim ∄.

$$\overset{?}{=} \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

$$\downarrow$$

$$0$$

let us check whether $\dfrac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2} \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2}$$

Along the path $\Delta x = \Delta y$: $\dfrac{\Delta x\, \Delta y}{\Delta x^2 + \Delta y^2} = \dfrac{\Delta x^2}{2\Delta x^2} = \dfrac{1}{2}$

Along the path $\Delta x = 0$: $\dfrac{\Delta x\, \Delta y}{\Delta x^2 + \Delta y^2} = 0$

Hence the limit ∄.

Hence $f(x,y)$ is not differentiable at $(0,0)$, although it has partial derivatives at $(0,0)$.

This is different from f-s of 1 var: $f(x)$ is diff. $\Leftrightarrow$ $f'(x)$ ∃.

Microsoft Whiteboard

$$\frac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2} \overset{?}{=} \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \to 0$$

$\lim \nexists$.

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

let us check whether $\dfrac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2} \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x \cdot \Delta y}{\Delta x^2 + \Delta y^2}$$

Along the path $\Delta x = \Delta y$: $\dfrac{\Delta x \, \Delta y}{\Delta x^2 + \Delta y^2} = \dfrac{\Delta x^2}{2\Delta x^2} = \dfrac{1}{2}$

Along the path $\Delta x = 0$: $\dfrac{\Delta x \, \Delta y}{\Delta x^2 + \Delta y^2} = 0$

Hence the limit $\nexists$.

Hence $f(x,y)$ is not differentiable at $(0,0)$, although it has partial derivatives at $(0,0)$.

This is different from f-s of 1 var: $f(x)$ is diff. $\iff f'(x) \, \exists$.

11:41 PM 9/7/2021

1 var: $f(x)$ is diff. $\Longleftrightarrow$ $f'(x)$ $\exists$.

2 var.: $f(x,y)$ is diff $\not\Longleftrightarrow$ f_x and f_y $\exists$.

<u>Sufficient</u> conditions for differentiability:

<u>Th.</u> If f_x and f_y are contin. near (a,b), then f is differentiable.

$$f(x,y) = \boxed{e^{\sin \frac{xy^3}{y-x}}}$$

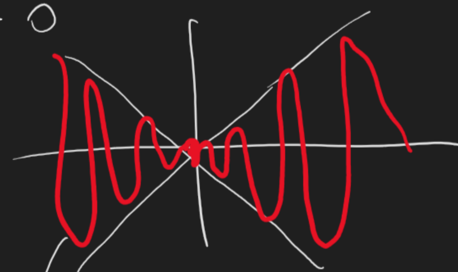
$$D = \{(x,y) \mid y \neq x\}.$$

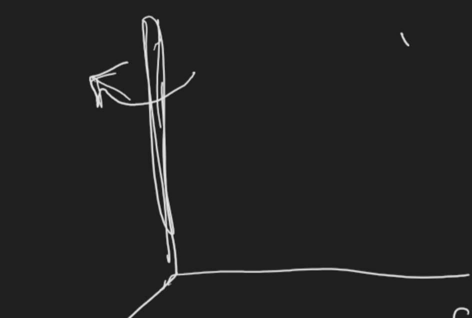
has contin. part. derivatives on D.

The Th. gives only sufficient conds. for diff-ty but not

The Th. gives only sufficient cond-s for diff.-ty, but not necessary. Here is an example.

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$





rotate this graph around vertical line

$$f(x,y) = \begin{cases} (x^2+y^2) \sin \frac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

f_x and f_y $\exists$, but f is not diff. at $(0,0)$.

Chain Rule.

Recall: $y = f(x)$, $x = g(t)$

(in other words we have f-

Microsoft Whiteboard

Chain Rule.

Recall: $y = f(x)$, $x = g(t)$

(in other words we have f-n $f(g(t))$)

$\frac{dy}{dt} = ?$ $\left(f(g(t))' = ? \right)$

Chain Rule for f-s of 1 var.: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

(equivalently, $f(g(t))' = f'(g(t)) \cdot g'(t)$)

Will be 2 versions of Chain Rule for f-s of 2 variables.

At first, suppose $z = f(x, y)$,

$x = g(t)$, $y = h(t)$.

So z is indirectly a f-n of 1 variable, t.

$\frac{dz}{dt} = ?$

$x = g(t)$, $y = h(t)$.

So z is indirectly a f-n of [illegible] t.

$$\frac{dz}{dt} = ?$$

Chain Rule (version 1): Suppose that $z = f(x, y)$ is differentiable f-n of x and y, where $x = g(t)$ and $y = h(t)$ are differentiable f-s of t. Then z is also differentiable and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Proof. Δt - increment in t. Let Δx be the corresponding increment in x,
Δy — — // — — in y.

(Namely $\boxed{\Delta x = g(t+\Delta t) - g(t).}$
$\Delta y = h(t+\Delta t) - h(t)$).

[illegible]t Δz be the corresponding increment in z.

Since z is a differentiable f-n,

$$\Delta z = f_x \, \Delta x + f_y \, \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

[illegible] ε_1 and $\varepsilon_2 \to 0$

$$\Delta z = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where ε_1 and $\dots$ as $(\Delta x, \Delta y) \to (0,0)$

Divide by Δt:

$$\underset{\downarrow\, \frac{dz}{dt}}{\frac{\Delta z}{\Delta t}} = f_x \underset{\downarrow\, \frac{dx}{dt}}{\frac{\Delta x}{\Delta t}} + f_y \underset{\downarrow\, \frac{dy}{dt}}{\frac{\Delta y}{\Delta t}} + \underset{\downarrow\, ?\to 0}{\varepsilon_1} \underset{\downarrow\, \frac{dx}{dt}}{\frac{\Delta x}{\Delta t}} + \underset{\downarrow\, ?\to 0}{\varepsilon_2} \underset{\downarrow\, \frac{dy}{dt}}{\frac{\Delta y}{\Delta t}}$$

When $\Delta t \to 0$, then $\Delta x \to 0$ because g is a contin. f-n and $\Delta x = g(t+\Delta t) - g(t)$.

When $\Delta t \to 0$, then $\Delta y \to 0$ because h is a contin. f-n and $\Delta y = h(t+\Delta t) - h(t)$

Hence $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta t \to 0$.

because $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

After taking $\lim\limits_{\Delta t \to 0}$, we obtain.

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}. \quad \square$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \quad \square$$

Since $z = f(x,y)$, one also somites writes

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

Ex. Let $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$. Find $\frac{dz}{dt}$ at $t = 0$.

Solution By Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} =$$

$$= (2xy + 3y^4)\cdot\cos(2t)\cdot 2 \; +$$

$$(x^2 + 12xy^3)\cdot(-\sin t)$$

Microsoft Whiteboard

$$= (2xy + 3y^4) \cdot \cos(2t) \cdot 2 +$$

$$(x^2 + 12xy^3) \cdot (-\sin t)$$

when $t = 0$, then $x = \sin(2 \cdot 0) = 0$

$y = \cos 0 = 1$.

Hence when $t = 0$,

$$\left.\frac{dz}{dt}\right|_{t=0} = 3 \cdot \cos(0) \cdot 2 = 6.$$

$\square$.

Now suppose $z = f(x, y)$,

$x = g(s, t)$, $y = h(s, t)$.

Then z is indirectly a f-n of s, t.

$$\boxed{\frac{\partial z}{\partial s}} = ? \qquad \frac{\partial z}{\partial t} = ?$$

$$z = f(x(s,t), y(s,t))$$

Computing $\frac{\partial z}{\partial s}$ we assume t is const

Hence we consider x and y as

Microsoft Whiteboard

Computing $\frac{\partial z}{\partial s}$ we assume t is const

Hence we consider x and y as

f-s of only 1 var., s. Then

Chain Rule (1 version) applies:

$$\frac{\partial z}{\partial s} \overset{\text{Chain Rule}}{=\!=} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

Similarly

$$\frac{\partial z}{\partial t} \overset{\text{Chain Rule}}{=\!=} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

hus we obtained

Chain Rule (version 2): to be continued.

11:43 PM 9/7/2021 17°C ENG