At Previous lecture:

Chain Rule (version 1): If z is a differentiable f-n of x and y, and x and y are differentiable f-s of t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Now consider such case:

z is a f-n of x and y,

x and y are f-s of 2 variables t and s.

$$\boxed{\frac{\partial z}{-}} = ? \qquad \frac{\partial z}{\partial s} = ?$$

Chain Rule (version 2): If z is a differentiable f-n of x and y, and x and y are differentiable f-s of t and s, then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t},$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

Ex If $z = f(x,y)$ has continuous

Ex. If $z = f(x,y)$ has continuous 2-order partial derivatives

$x = r^2 + s^2$ and $y = 2rs$, find

a) $\frac{\partial z}{\partial r}$, b) $\frac{\partial^2 z}{\partial r^2}$.

Solution a) $\frac{\partial z}{\partial r}$ Chain Rule (vers. 2)

$$= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} =$$

$$= \boxed{\frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial y}\cdot 2s}$$

b) $$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial y}\cdot 2s\right) =$$

(Since z has contin. 2^{nd} order part. deriv., then $\frac{\partial z}{\partial r}$ has contin. part. deriv., hence it is differentiable and Chain Rule applies to it)

$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot 2r\right) + \boxed{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\cdot 2s\right)} =$$

Product Rule

$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cdot 2r + \frac{\partial z}{\partial x}\cdot 2 + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\cdot 2s =$$

Chain Rule

$$2r\left(\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\cdot\frac{\partial y}{\partial r}\right) + \frac{\partial z}{\partial x}\cdot 2 +$$

$$+ 2s\cdot\left(\frac{\partial^2 z}{\partial x\partial y}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial^2 y}\cdot\frac{\partial y}{\partial r}\right) =$$

$$= 2r\left(\frac{\partial^2 z}{\partial x^2}\cdot 2r + \frac{\partial^2 z}{\partial^2 y}\cdot 2s\right) + 2\frac{\partial z}{\partial x} +$$

$$+ 2s\left(\frac{\partial^2 z}{\partial x\partial y}\cdot 2r + \frac{\partial^2 z}{\partial^2 y}\cdot 2s\right).$$ □

Now the most general case:

u depends on $x_1, x_2, \ldots, x_n$,

and each x_i depends on the m

variables $t_1, \ldots, t_m$.

$\frac{\partial u}{\partial t_i} = ?$ By the same argument as for Chain Rule (version 2)

we get

Chain Rule (general version) Suppose u is a differentiable f-n of $x_1, \ldots, x_n$, and each x_i is a differentiable f-n of $t_1, \ldots, t_m$.

Then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Ex. Let $u = x^4 y + y^2 z^3$, where

$x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 + s\sin t$

Find the value of $\frac{\partial u}{\partial s}$ when

$r = 2,\ s = 1,\ t = 0$.

Solution $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$

$= 4x^3y \cdot re^t + (x^4 + 2yz^3) \cdot 2rse^{-t} + 3y^2z^2 \cdot \sin t$

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Implicit Differentiation

$y(x)$ – a f-n of 1 variable.

Sometimes a f-n is given implicitly (for example $\boxed{x^3+3xy-y^5+1=0}$)

One often needs to find $\frac{dy}{dx}=?$

Would be great to express y from the equation, but it is not always possible.

Introduce the f-n of 2 variables

$$F(x,y):=x^3+3xy-y^5+1$$

If y is our f-n, then $\boxed{F(x,y)=0}$ $(*)$.

F depends of x and y, y depends on x, x depends on x. Apply Chain Rule (version 1) to differentiate $(*)$:

$$\frac{\partial F}{\partial x}\underbrace{\boxed{\frac{dx}{dx}}}_{=1}+\frac{\partial F}{\partial y}\boxed{\frac{dy}{dx}}=0$$

what we want to find.

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

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$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

Since we divided by F_y, we should assume it is $\neq 0$.

Thus we obtained such rule:

~~where F is a differentiable f-n~~
If y is a f-n of x given implicitly by an equation of the form $F(x,y)=0$, and $F_y \neq 0$, then

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

Let us finish our example:

Ex. $y(x)$ is given implicitly by $x^3 + 3xy - y^5 + 1 = 0$. Find $\frac{dy}{dx}$ at $x = 0$.

Solution $F(x,y) := x^3 + 3xy - y^5 + 1$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{-\frac{3x^2 + 3y}{3x - 5y^4}}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{3}{-5} = \frac{3}{5}. \quad \square$$

when $x=0$,

$-y^5 = \ldots$

$y^5 = 1$

$y = 1$

Now suppose z is a f-n of $\underline{x, y}$ given implicitly by an equation

$$\boxed{F(x, y, z) = 0}.$$

Can apply Chain Rule (version 2) to diff. the equation w.r.t. (with respect to) x and w.r.t. y.

w.r.t. x:

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1 = \frac{dx}{dx}} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{\text{what we want to find.}} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}}$$

Differentiating with respect to y, we obtain

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

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Ex. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

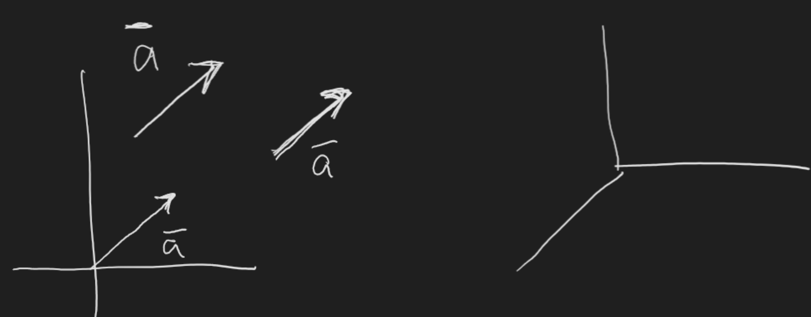
$$x^3+y^3+z^3+6xyz=1.$$

Solution. $F(x,y,z) := x^3+y^3+z^3+6xyz-1.$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2+6yz}{3z^2+6xy}.$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy}. \quad \square$$

Recall: vectors.



A vector has direction and length.

$\bar{a}$ — means vector

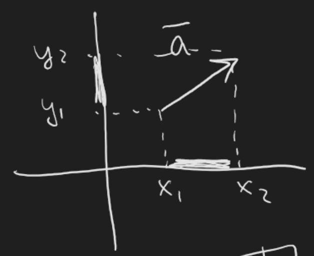
(don't confuse with

complex conjugation).

$\bar{a} = \langle a, b \rangle$ or $\bar{a} = \langle a_1, a_2, a_3 \rangle$.

length:

$$|\bar{a}| = \sqrt{a^2 + b^2}$$

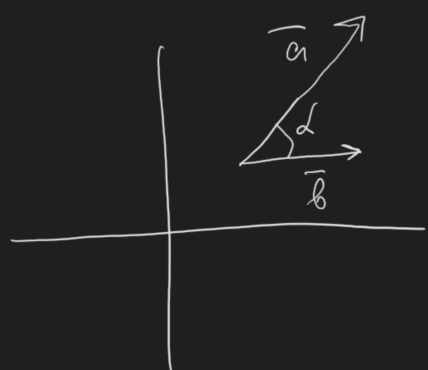


$\bar{a} = \langle a, b \rangle$ Then
$a = x_2 - x_1$, $b = y_2 - y_1$.

Dot product (scalar product, inner product).

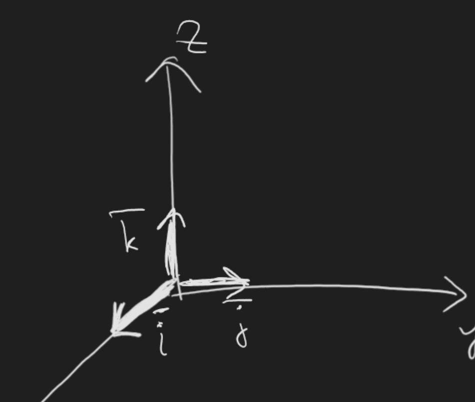
$\bar{a} = \langle a_1, a_2, a_3 \rangle$, $\bar{b} = \langle b_1, b_2, b_3 \rangle$

$$\bar{a} \cdot \bar{b} := a_1 b_1 + a_2 b_2 + a_3 b_3$$



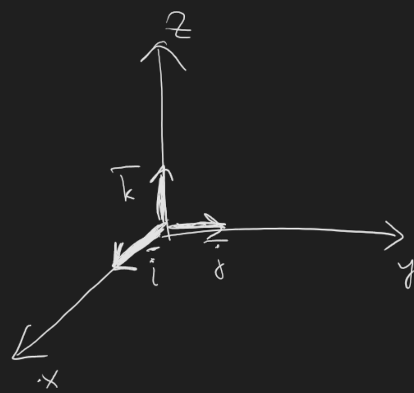
<u>Theorem:</u> $\bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cdot \cos \alpha$

Standard basis vectors:



Theorem: $\bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cdot \cos \alpha$

Standard basis vectors:



$\bar{i} = \langle 1, 0, 0 \rangle$

$\bar{j} = \langle 0, 1, 0 \rangle$

$\bar{k} = \langle 0, 0, 1 \rangle.$