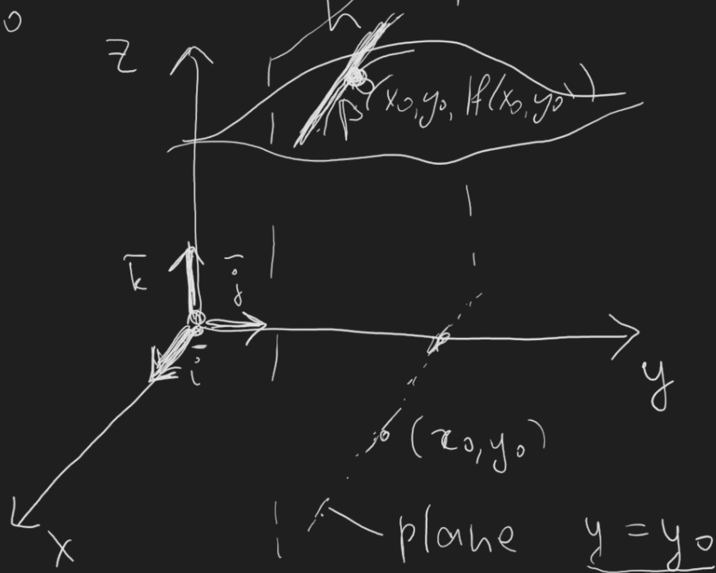
Directional derivative

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Geom. sense:



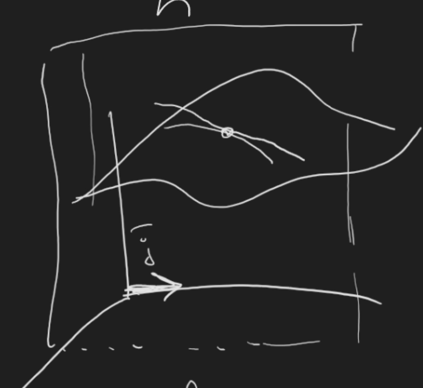
(vertical plane going through P in the direction of $\bar{i}$).

Also one can say:

$f_x(x_0, y_0)$ shows how much the f-n changes in the direction of $\bar{i}$, or in other words,

the rate of change of f in the direction of $\bar{i}$.

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$



$f_y(x_0, y_0)$ shows the rate of change of f in the

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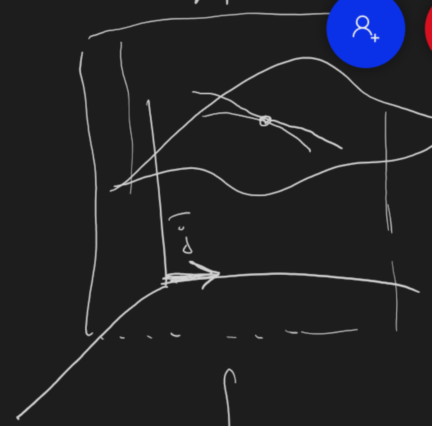
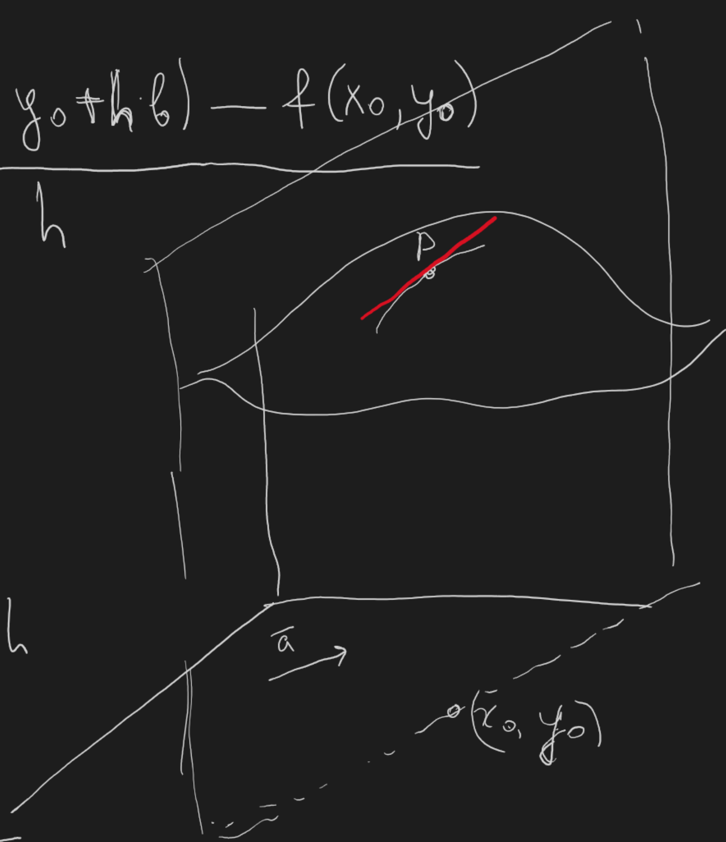


$f_y(x_0, y_0)$ shows the rate of change of f in the direction of $\bar{j}$.

unit vector

Take $\bar{a} = \langle a, b \rangle$ and consider

$$\lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$



eom. sense of
his limit:

ntersect the graph
y the vertical
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You get a curve and look at the tangent line at P. Its slope is the limit above.

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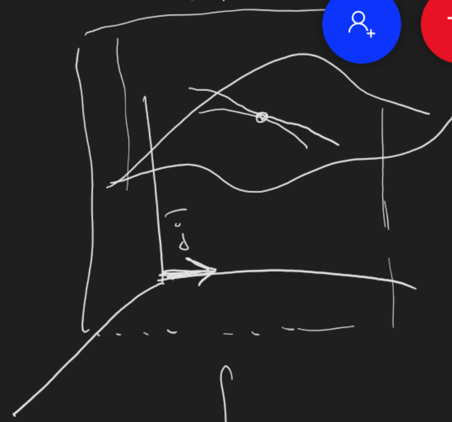

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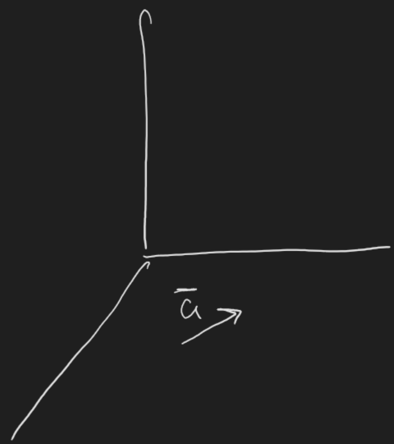

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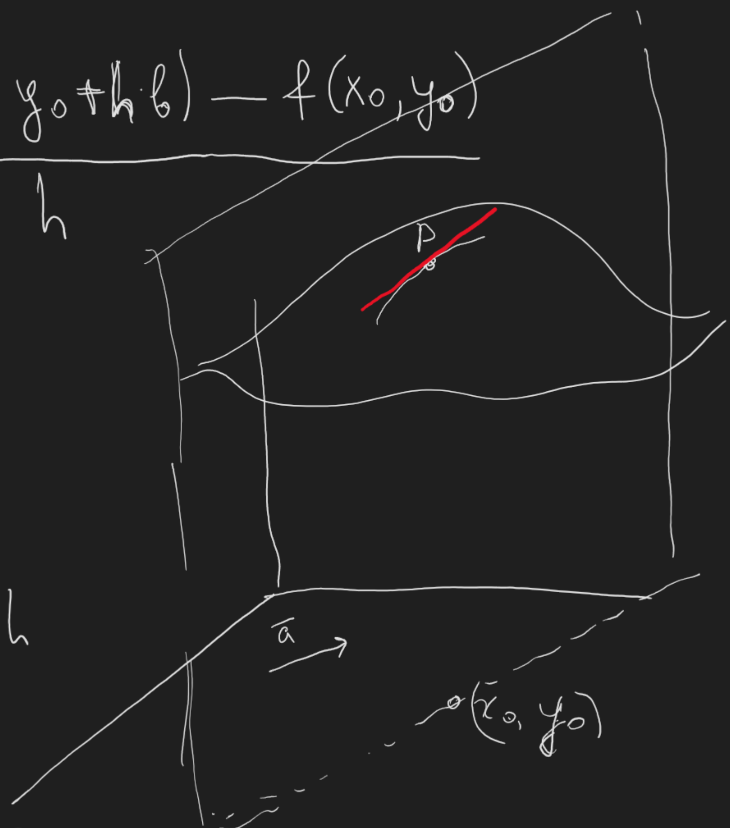


Geom. sense of this limit:

Intersect the graph by the vertical plane going through P in the direction of $\bar{a}$.

You get a curve and look at the tangent line at P. Its slope is the limit above.

the rate of change of f in the direction of $\bar{a}$.

Def. The directional derivative of f at point (x_0, y_0) in the direction of the unit vector $\bar{u} = \langle a, b \rangle$ is the limit

$$D_{\bar{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

---

f_x and f_y are particular cases of directional derivatives.

How to compute $D_{\bar{u}} f$?

Theorem. Suppose f is a differentiable f-n of variables x and y. Then there exist the directional derivative $D_{\bar{u}} f$ in the direction of any unit vector $\bar{u} = \langle a, b \rangle$ and

$$D_{\bar{u}} f = f_x \cdot a + f_y \cdot b.$$

D f Want: $D_{\bar{u}} f \overset{?}{=} f_x a + f_y b$

Microsoft Whiteboard

Proof. Want: $D_u f \stackrel{?}{=} f_x a + f_y b$

Take any (x_0, y_0).

Want: $D_u f(x_0,y_0) \stackrel{?}{=} f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b.$

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{\overbrace{f(x_0+ha, y_0+hb)}^{g(h)} - f(x_0,y_0)}{h}$$

let $\boxed{g(h) := f(\underbrace{x_0+ha}_{x}, \underbrace{y_0+hb}_{y})}$

$\boxed{g(0) = f(x_0,y_0)}$

Then $D_u f(x_0,y_0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = g'(0).$

On the other hand, $g(h) = f(x,y)$,

where $x = x_0 + h\cdot a$, $y = y_0 + h\cdot b$.

By Chain Rule we can differentiate g:

$$g'(h) = f_x \cdot \underbrace{\frac{dx}{dh}}_{=a} + f_y \cdot \underbrace{\frac{dy}{dh}}_{=b} = f_x^{(x,y)}\cdot a + f_y^{(x,y)}\cdot b.$$

$$D_{\bar u} f(x_0,y_0) = g'(0) = f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b$$

$\bar u = \langle a, b\rangle$ – unit vector.

$$D_{\bar u} f = f_x \cdot a + f_y \cdot b =$$

$$= \langle f_x, f_y\rangle \cdot \langle a, b\rangle$$

Poll: which name of this product are you used to:

a) dot product
b) inner product
c) scalar product

$\langle f_x, f_y\rangle$ is called gradient of f.

vector-function (meaning a vector whose coordinates are f-s).

Def. The gradient of f is

the vector-f-n

Def. The gradient of f is the vector-f-n

$$\nabla f = \langle f_x, f_y \rangle.$$

Hence

$$D_{\bar{u}} f = \underbrace{\langle f_x, f_y \rangle}_{\nabla f} \cdot \underbrace{\langle a, b \rangle}_{\bar{u}} = \nabla f \cdot \bar{u}.$$

$$\boxed{D_u f = \nabla f \cdot \bar{u}}$$

Ex. Find the directional derivative of the f-n $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector

a) $\bar{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

b) $\bar{v} = 2\bar{i} + 5\bar{j}$

Solution

$$D_{\bar{v}} f = \nabla f \cdot \bar{v},$$ when $\bar{v}$ is a unit vector.

$\ldots\ 3 \ldots 3x^2y^2 - 4 \rangle$

Solution $D_{\bar v} f = \nabla f \cdot \bar v$,

$$\nabla f = \langle f_x, f_y \rangle = \langle 2xy^3, 3x^2y^2 - 4\rangle$$

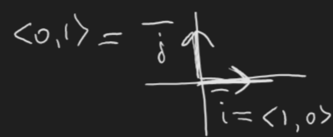
Could find here $\nabla f(2,-1)$ to not do it later.

a) $|\bar v| = \left|\left\langle \frac{1}{\sqrt2}, \frac{1}{\sqrt2} \right\rangle\right| = \sqrt{\frac12 + \frac12} = 1$

So $\bar v$ is a unit vector.

$$D_{\bar v} f = \nabla f \cdot \bar v = \langle 2xy^3, 3x^2y^2 - 4\rangle \cdot \left\langle \frac{1}{\sqrt2}, \frac{1}{\sqrt2} \right\rangle = \frac{2xy^3}{\sqrt2} + \frac{3x^2y^2 - 4}{\sqrt2}$$

$$D_{\bar v} f(2,-1) = \frac{2\cdot 2\cdot(-1)^3}{\sqrt2} + \frac{3\cdot 2^2(-1)^2 - 4}{\sqrt2} = \dots$$

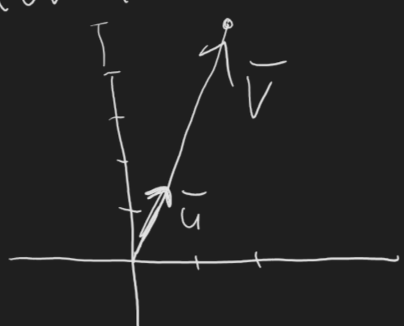


b) $\bar v = 2\bar i + 5\bar j = 2\langle 1,0\rangle + 5\langle 0,1\rangle = \langle 2,5\rangle$

$$|\bar v| = \sqrt{2^2 + 5^2} = \sqrt{4+25} = \sqrt{29} \neq 1.$$

So $\bar v$ is not a unit vector.

Let $\bar u$ be the unit vector in the direction of $\bar v$:



$$\bar u = \frac{\bar v}{|\bar v|} = \frac{\langle 2,5\rangle}{\sqrt{29}} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

The directional derivative of f in the direction of $\bar v$ is

$$\bar{u} = \frac{\bar{v}}{|\bar{v}|} = \frac{\langle 2, 5\rangle}{\sqrt{29}} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

The directional derivative of f in the direction of $\bar{v}$ is

$$D_{\bar{u}} f = \nabla f \cdot \bar{u} =$$

$$= \langle 2xy^3, 3x^2y^2 - 4\rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle =$$

$$= \frac{4xy^3}{\sqrt{29}} + \frac{5(3x^2y^2 - 4)}{\sqrt{29}}.$$

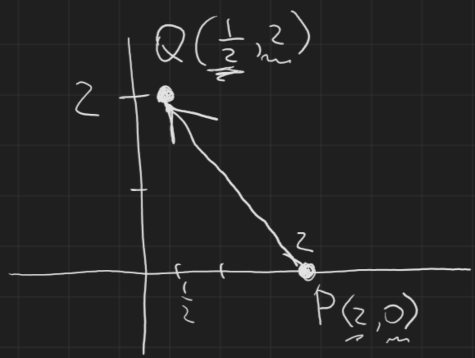
$$D_u f(2, -1) = \frac{4 \cdot 2 \cdot (-1)^3}{\sqrt{29}} + \frac{5(3 \cdot 2^2 \cdot (-1)^2 - 4)}{\sqrt{29}} =$$

$$= \ldots$$

Ex. Find the rate of change of the f-n $f(x,y) = xy^2 + 1$ at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

Solution Rate of change of f in the direction... = the directional derivative.

Solution: Rate of change of f in the direction ... = the directional derivative.



We need to find the directional derivative of f in the direction of the vector $\vec{PQ}$.

$$\vec{PQ} = \left\langle \tfrac{1}{2}-2,\ 2-0 \right\rangle = \left\langle -\tfrac{3}{2},\ 2 \right\rangle.$$

$$|\vec{PQ}| = \sqrt{\tfrac{9}{4}+4} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2}. \neq 1$$

$\vec{PQ}$ is not a unit vector.

$$\bar{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\left\langle -\frac{3}{2}, 2 \right\rangle}{\frac{5}{2}} = \left\langle -\tfrac{3}{5}, \tfrac{4}{5} \right\rangle$$

– the unit vector in the same direction as $\vec{PQ}$.

$$D_{\bar{u}} f = \nabla f \cdot \bar{u}.$$

$$\nabla f = \langle f_x, f_y \rangle = \langle y^2, 2xy \rangle.$$

$$\nabla f(2,0) = \langle 0, 0 \rangle.$$

$$D_{\bar{u}} f(2,0) = \langle 0,0 \rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5} \right\rangle = 0.$$

$\square$

In which direction the rate of change of a f-n is the largest?

Turns out, in the direction of ∇f.

Suppose f is differentiable, $\bar u$ is a unit vector.

__Theorem__ $\boxed{D_{\bar u} f(x_0, y_0)}$ is the largest if $\bar u$ has the same direction as $\nabla f(x_0, y_0)$.

In this case the value of $D_{\bar u} f(x_0, y_0)$ equals to $\underbrace{|\nabla f(x_0, y_0)|}_{\text{the length of } \nabla f(x_0, y_0)}$.

__Proof__ $D_{\bar u} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \bar u =$

Recall $\bar v \cdot \bar w = |\bar v| \cdot |\bar w| \cdot \cos \alpha$.

$= |\nabla f(x_0, y_0)| \cdot \overset{=1}{|\bar u|} \cdot \cos \alpha$, where

α is the angle between $\nabla f(x_0, y_0)$ and $\bar u$.

α is the angle between $\nabla f(x_0,y_0)$ and u.

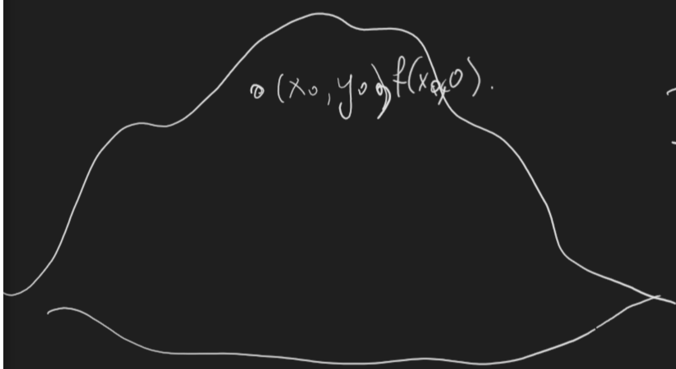
$$= |\nabla f(x_0,y_0)| \cdot \cos\alpha \leq |\nabla f(x_0,y_0)|.$$

$$D_{\bar{u}} f(x_0,y_0) = \boxed{|\nabla f(x_0,y_0)|} \quad \text{when } \cos\alpha = 1$$

$$\alpha = 0.$$

$\xrightarrow{\bar{u}}$ $\xrightarrow{\nabla f(x_0,y_0)}$

Thus $D_{\bar{u}} f(x_0,y_0)$ is the largest when $\bar{u}$ and $\nabla f(x_0,y_0)$ have the same direction. $\square$.



<u>Interpretation:</u>

graph of f — mountain,

point — where you stay.

You want to go down the mountain as fast as possible. (meaning, choose the steepest way).

Theorem says, you should go in the direction of the gradient.

<u>Ex.</u> For $f(x,y) = xe^{y}$ and $P(2,0)$

Ex. For $f(x,y) = xe^{y}$ and $P(2,0)$

a) in what direction f has maximal rate of change?

b) what is this maximal rate of change?

Solution a) in the direction of

$$\nabla f(2,0) = \langle f_x(2,0), f_y(2,0) \rangle =$$

$$= \langle e^{y}\big|_{(2,0)}, \; xe^{y}\big|_{(2,0)} \rangle = \langle 1, 2 \rangle.$$

b) $|\nabla f(2,0)| = |\langle 1,2 \rangle| = \sqrt{1^2+2^2} = \sqrt{5}.$ $\square$