At previous lecture:

gradient $\nabla f = \langle f_x, f_y \rangle$

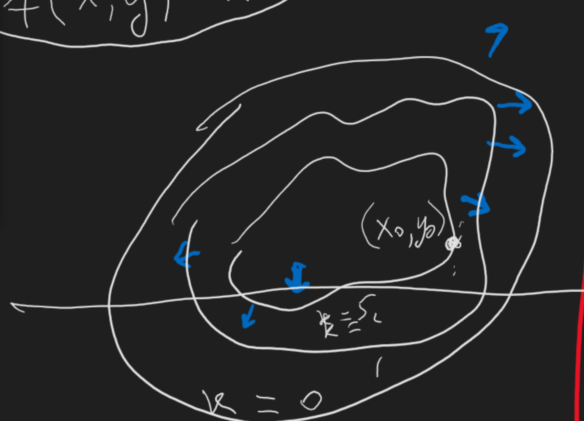
directional derivative of f in the directio[n] of a unit vector $\bar{u} = \langle a, b \rangle$

$$D_{\bar{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}.$$

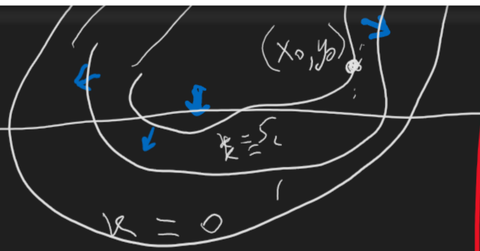
Theorem: $D_{\bar{u}} f = \nabla f \cdot \bar{u}$.

Gradient and Level curves

$f(x,y) = k$



$\boxed{\nabla f(x_0,y_0) \text{ is orthogonal to the level curve at this point.}}$



to the level curve at this point.

Intuitive reasoning: we already know that the fastest way down the mountain is to go in the direction of ∇f. In terms of level curves, the ~~fastest~~ means to go from level curve to level curve with smaller height as fast as possible.
On the other hand, the fastest way is to go perpendicularly. Hence ∇f is perpendicular to level curves.

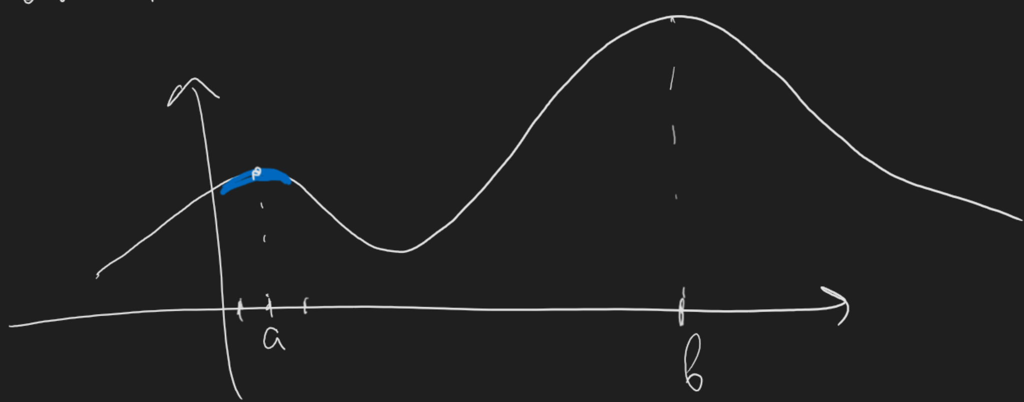
For f-s of 3 or more variables everything is similar.

$$\nabla f = \langle f_x, f_y, f_z \rangle, \quad \bar{u} = \langle a, b, c \rangle$$

$$D_{\bar{u}} f = \nabla f \cdot \bar{u} \quad \ldots$$

Maximum and Minimum Values.

For f-s of 1 variable



At $x = b$ f has absolute (or global) max.

At $x = a$ (at $x = b$) f has local max

(It means $f(a) \geq f(x)$ for all pts x in some interval centered at a).

Def. A f-n of 2 variables has a local maximum at (a, b) if

$f(x,y) \leq f(a,b)$ for all points (x,y) near (a,b)

[This means $f(x,y) \leq f(a,b)$ for all points (x,y) from some disc centered at (a,b)].

[This means $f(x,y) \le f(a,b)$ for all points (x,y) from some disc centered at (a,b)].

f has <u>local minimum</u> at (a,b) if

$f(x,y) \ge f(a,b)$ for all (x,y) near (a,b)

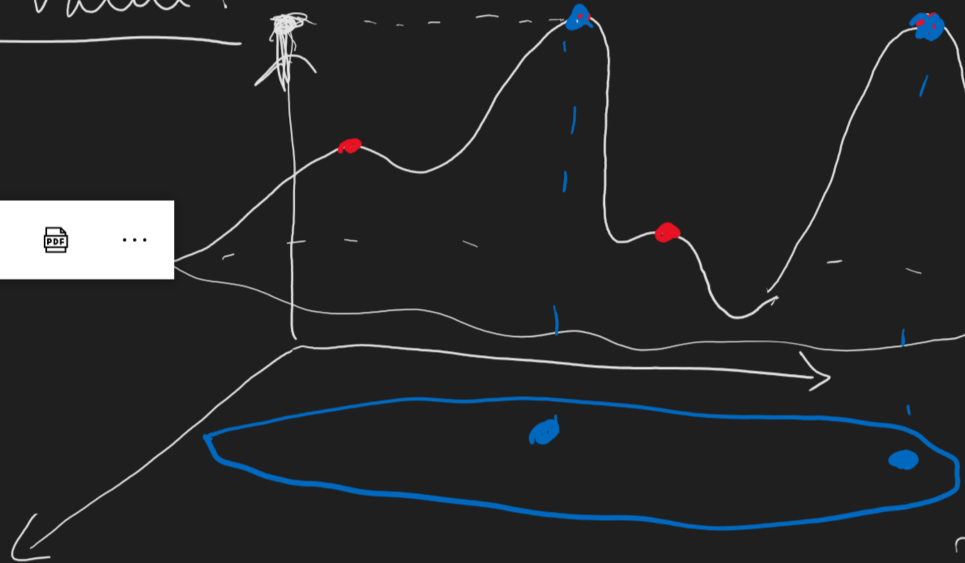
$f(a,b)$ is called the local max/min value.

(We also says that (a,b) is a point of local max/min.)

<u>Def.</u> If $f(x,y) \le f(a,b)$ for all (x,y) in the domain of f, then f has an <u>absolute maximum</u> at (a,b).
$f(a,b)$ is the <u>absolute maximum value</u>.

If $f(x,y) \ge f(a,b)$ for all (x,y) in the domain of f, then f has <u>abs. minimum</u> at (a,b). $f(a,b)$ is <u>the absolute</u> minimum value at (a, b). $f(a, b)$ is the absolute minimum value.

Po



How many abs. max values does f have?

a) 2

b) 1

c) 4

How to find loc. max/min ?

Recall: f-n of 1 var.

If a is a point of loc. max/min and if $f'(a)$ $\exists$, then

$$f'(a) = 0.$$

Geometrically:

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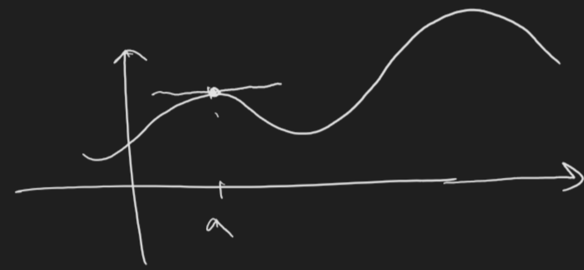
Geometrically:



at pts of loc. max/min the tangent line is horizontal.

Theorem. If f has a loc. max/min at (a, b) and if the partial derivatives exist at (a, b), then

$$f_x(a, b) = 0, \quad f_y(a, b) = 0$$

(Equivalently, $\nabla f(a, b) = 0$).

Proof. (Idea is to reduce to the case of f-n of 1 variable)

Say, f has a loc. max at (a, b).

$$f(x, y) \le f(a, b) \quad \text{when } (x, y) \text{ is near } (a, b)$$

So, f has a loc. max at (a,b)

$$f(x,y) \le f(a,b) \quad \text{when } (x,y) \text{ is near } (a,b).$$

In particular, $\underbrace{f(x,b)}_{=\, g(x)} \le \underbrace{f(a,b)}_{=\, g(a)}$ when x is near a.

$g(x) := \boxed{f(x,b)}$ — a f-n of 1 var.

$g(x) \le g(a)$ when x is near a.

g has loc. max at a.

Hence $g'(a) = 0$.

$$\| \quad f_x(a,b).$$

Similarly, fixing $x=a$ and considering $g(y) = f(a,y)$ we would obtain $f_y(a,b) = 0$. $\square$

Geometrical interpretation:

Geometrical interpretation:

tangent line at pts of loc. max/min is horizontal.

Def. (a, b) is a critical point (or a stationary point) if either $\nabla f(a,b) = 0$ or $\nabla f(a,b) \nexists$.

Therefore, Theorem says:

if (a,b) is a point of loc. max/min, then it is a critical point.

⚠ Not the other way around!

Microsoft Whiteboard

Therefore, Theorem says:

if (a,b) is a point of loc. max/min, then it is a critical point.

Not the other way around! at a critical point a f-n might not have loc. min/max.

Ex. Find the critical points of the f-n $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

Solution. $\nabla f = \langle 2x-2, 2y-6 \rangle = 0$

$2x - 2 = 0$ and $2y - 6 = 0$

$x = 1$ and $y = 3$.

So we have one critical pt: $(1,3)$. $\square$.

How to see whether at a critical pt we have loc. max or loc. min or neither?

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How to see whether at a critical pt we have loc. max or loc. min or neither?

Need to look at 2^{nd} order partial derivatives!

Recall: f is a f-n of 1 var. and a is a critical pt (so $f'(a)=0$).

Then if $f''(a) > 0 \implies$ loc. min.

$f''(a) < 0 \implies$ loc. max.

$(x^2)'' = 2.$

2^{nd} Derivative Test:

Suppose the 2^{nd} order partial derivatives of f are continuous on a disc centered at (a,b) and suppose (a,b) is a critical point, Microsoft Whiteboard

TS

2nd Derivative Test: suppose the 2nd order partial derivatives of f are continuous on a disc centered at (a,b) and suppose (a,b) is a critical point,

let $D \equiv D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$

a) If $D>0$ and $\boxed{f_{xx}(a,b)}>0$, then there is a loc. min at (a,b).

b) If $D>0$ and $f_{xx}(a,b)<0$, then there is a loc. max at (a,b).

c) If $D<0$, then there is neither loc. max nor loc. min at (a,b).

Remark 1. In case (c), (a,b) is called a saddle point.

Remark 2 The test does not give

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has loc. min at (a, b).

Remark 1. In case (c), (a, b) is called a saddle point.

emark 2 The test does not give information what we have when $D = 0$.

emark 3 $D = \boxed{f_{xx} f_{yy} - (f_{xy})^2} = \boxed{\det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}}$

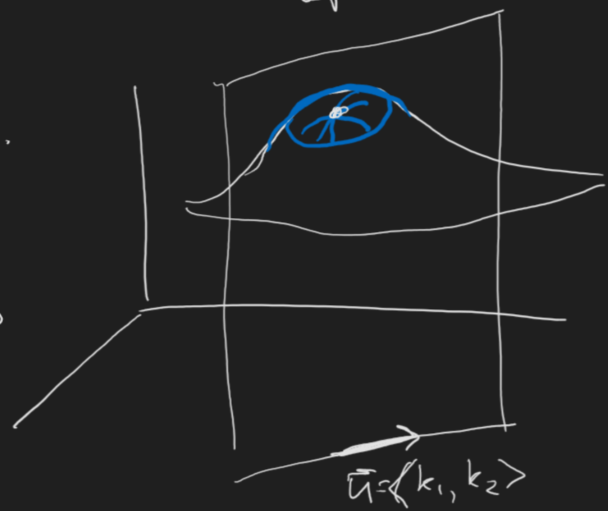
– it might be an easier way to remember D.

a of proof (only idea, no details)

educe to the case of f-s of 1 var. and then use 2^{nd} Der. Test for f-s of 1 var.

… reduce to the case of f-s of 1 var. and then use 2^{nd} Der. Test … s of 1 var.

(a,b) – a pt of loc. max.



(a,b) is loc. max $\iff$ (a,b) is a loc. max at …ny curve obtained by …ntersection of the graph by any vertical plane.

…ut a curve is given by a f-n of 1 var.

…gebraically: consider a f-n

$$g(h) := f(a+k_1 h,\ b+k_2 h).$$

– a f-n of 1 var.

its graph is

$$\boxed{D_{\bar u}\left(D_{\bar u} f\right)} = g'' \begin{array}{l} > 0 \Rightarrow \text{loc. min} \\ < 0 \Rightarrow \text{loc. max.} \end{array}$$

arguments were given in the proof of the fact $D_{\bar u} f = \nabla f \cdot \bar u$.

So one comes to exploring when

$$D_{\bar u}\left(D_{\bar u} f\right) > 0$$

so one comes to exploring

$$D_{\bar u}(D_{\bar u} f) \begin{matrix} > 0 \\ < 0. \end{matrix}$$

If one writes out $D_{\bar u}(D_{\bar u} f)$ in terms of f_{xx}, f_{xy}, f_{yy}, then the expression $D = f_{xx} f_{yy} - (f_{xy})^2$ arises there. □

Ex. Find loc. max, loc. min and saddle pts of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution. 1 step: Find critical pts.

$$\nabla f = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle = 0.$$

$4x^3 = 4y$ $\qquad$ $4y^3 = 4x.$

$x^3 = y$ $\leftarrow$ substitute $\qquad$ $y^3 = x.$

$$y^9 = y$$

$$y(y^8 - 1) = 0$$

either $y = 0$ or $y^8 = 1$

$y = 1$ or -1.

$y \equiv 0,\quad x = 0$

$y = 1$ or -1.

$y = 0, \quad x = 0$

$y = 1, \quad x = 1$

$y = -1, \quad x = -1.$

3 critical pts: $\boxed{(0,0), (1,1), (-1,-1).}$

step. Compute 2^{nd} order part. derivatives and D.

$f_x = 4x^3 - 4y, \quad f_y = 4y^3 - 4x.$

$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = -4.$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16 =$

$= 16(9x^2y^2 - 1).$

step. We classify our critical pts.

$(0,0) = -16 < 0 \xrightarrow{2^{nd}\text{ Der. Test (c)}} (0,0)$ is a saddle point.

$D(1,1) = 16(9-1) = 16 \cdot 8 > 0.$

$f_{xx}(1,1) = 12 > 0$

$\xrightarrow{2^{nd}\text{ Der. Test (a)}}$ $(1,1)$ is a point of local min.

saddle point.

$D(1,1) = 16(9-1) = 16 \cdot 8 > 0.$

$f_{xx}(1,1) = 12 > 0$

$\xRightarrow{2^{nd}\text{ Der. Test (a)}}$ $(1,1)$ is a point of local min.

$f(1,1) = -1$ – the local min value.

$D(-1,-1) = 16 \cdot 8 > 0.$

$f_{xx}(-1,-1) = 12 > 0$

$\Longrightarrow$ $(-1,-1)$ is a point of local minimum.

$f(-1,-1) = -1$ – the local min value.

$\square$