<u>At previous lecture:</u>

To find loc. max/min we look at critical pts

(a pt (x_0, y_0) is critical if either $\nabla f(x_0, y_0) = 0$ or $\nabla f(x_0, y_0)$ $\nexists$).

<u>2^{nd} Derivative Test</u>: if (x_0, y_0) is a critical pt, we should look at

$$D = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$$

(a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0 \implies$ loc. min

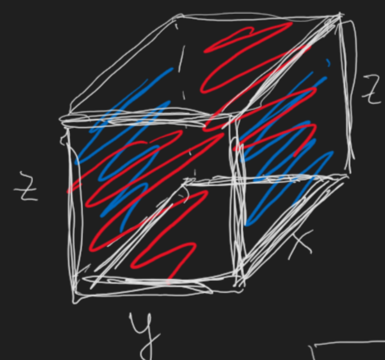
(b) if $D > 0$ and $f_{xx}(x_0, y_0) < 0 \implies$ loc. max

(c) if $D < 0 \implies$ saddle pt.

<u>Ex.</u> A rectangular box without a lid is to be made from $12 m^2$ of cardboard. Find a maximum volume of such a box

Ex. A rectangular box without lid is to be made from $12\,m^2$ of cardboard. Find a maximum volume of such a box.

Solution



$$\boxed{V = xyz}$$

Area of walls is

$$\boxed{2zx + 2zy + yx = 12}$$

Express z: $z(2x+2y) + yx = 12$

$$z = \frac{12 - yx}{2x+2y} = \frac{12 - yx}{2(x+y)}$$

Substitute into V:

$$\boxed{V = xy \cdot \frac{12 - yx}{2(x+y)}} \rightsquigarrow \text{abs. max}$$

$$V(2,2) = 4 \cdot \frac{12-4}{2(2+2)} = 4 \cdot \frac{8}{8} = 4$$

$$\nabla V = \left\langle \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2}, \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2} \right\rangle = 0$$

$\boxed{x, y > 0}$

$$12 - 2xy - x = 0,$$
$$12 - 2xy - y^2 = 0 \implies$$

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$2(x+y)^2$ $2(x+y)^2$

$x, y > 0$

$12 - 2xy - x = 0$,

$12 - 2xy - y^2 = 0$ $\Longrightarrow$

$y^2 - x^2 = 0$

$\boxed{y = x}$

Substitute into:

$12 - 2x^2 - x^2 = 0$

$12 = 3x^2$

$x^2 = 4$

$x = 2$, $y = 2$.

We got 1 critical pt $\boxed{(2, 2)}$.

In this problem abs. max must exist (because you cannot get arbitrary big volume out of $12\,m^2$ of cardboard).

Hence $(2,2)$ is a pt of abs. max.

Abs. max value $V(2,2) = 4$

□

In general, do abs. max/abs. min exist?

Al... ...d minimum

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In general, do abs. max/abs. min exist?

Abs. maximum and minimum.

(Abs. max and min are called extreme values)

For f-s of 1 variable:

Extreme Value Theorem If f is a contin. f-n of 1 variable on [a, b], then f has abs. max and abs. min.

On (a, b) — no

On $\mathbb{R}$ — no

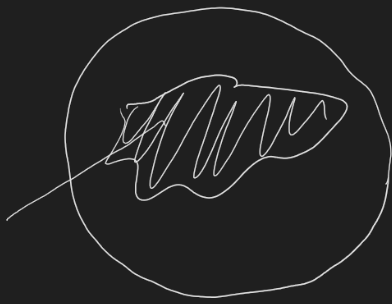
Def.

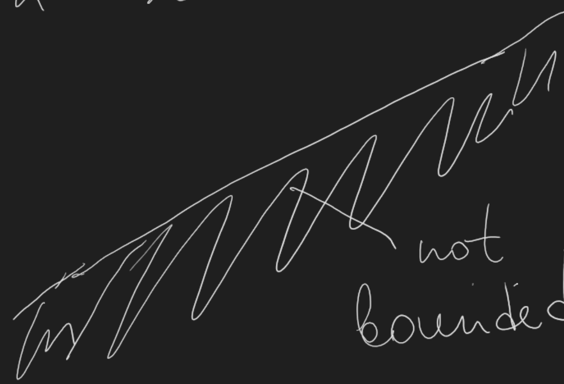
A set in $\mathbb{R}^2$ is bounded if

Def.

A set in $\mathbb{R}^2$ is <u>bounded</u> if it is contained in some disc.

bounded

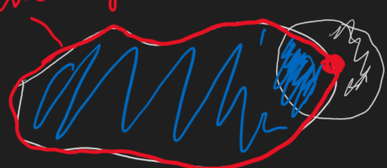




not bounded.

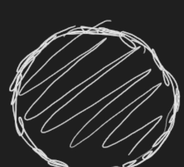
A set in $\mathbb{R}^2$ is <u>closed</u> if it contains all the boundary pts.

boundary



A pt in set D is called a <u>boundary pt</u> if any disc centered in it contains both pts from D and pts from outside D.

<u>Examples:</u>

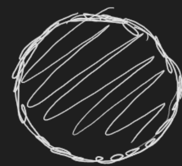


— closed.

$\{(x,y) \mid x^2+y^2 \le 1\}$

— not closed.

Examples:



— clo d

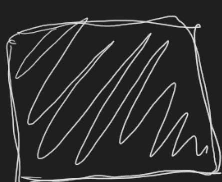
$\{(x,y) \mid x^2+y^2 \le 1\}$

— not closed.

$\{(x,y) \mid x^2+y^2 < 1\}$

— not closed.

(a disc with 1 point on the circle removed)



closed

not closed.

The following theorem is proved in Topology.

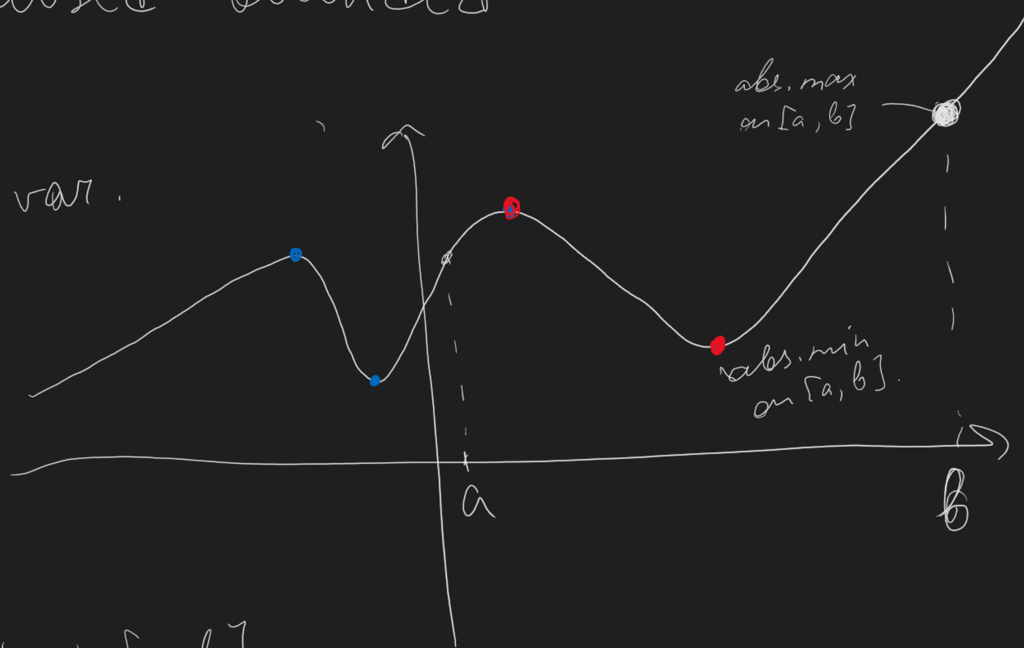
Extreme Value Theorem If f is a contin f-n of 2 variables on a bounded

Extreme Value Theorem

If f is a contin f-n of 2 variables on a bounded closed set, then f has the abs. max and min values.

How to find abs max/min on a closed bounded set?

f-n of 1 var.



abs max/min on $[a, b]$

To find abs. max/min of a contin. f-n f on a closed bounded set R

1 step Find critical pts in R and compute value of f at them.

2 step Find abs max/min on the boundary of f.

3 step The largest of the values found is

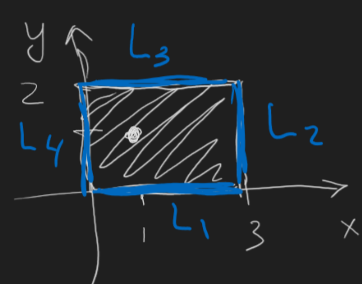
value of f at them.

2 step Find abs max/min on the boundary of f.

3 step The largest of the values found is steps 1 and 2 is the abs. max.
The smallest is the abs. min.

Ex. Find the abs. max and min values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $R = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution



R is closed and bounded
$\Rightarrow$ abs. max/min exist on R.

Step 1 $\nabla f = \langle 2x - 2y,\ -2x + 2 \rangle = 0$

$2x - 2y = 0, \quad -2x + 2 = 0$

$\Rightarrow x = y, \quad x = 1.$

$(1,1)$ is a critical pt in D.

$f(1,1) = 1^2 - 2\cdot 1\cdot 1 + 2\cdot 1 = 1$

Step 2 Boundary $= L_1 \cup L_2 \cup L_3 \cup L_4$

$L_1 = \{(x,y) \mid 0 \le x \le 3,\ y = 0\}$

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$L_1 = \{(x,y) \mid 0 \le x \le 3,\ y = 0\}$.

$L_2 = \{(x,y) \mid x = 3,\ 0 \le y \le 2\}$.

$L_3 = \{(x,y) \mid 0 \le x \le 3,\ y = 2\}$

$L_4 = \{(x,y) \mid x = 0,\ 0 \le y \le 2\}$.

$$\boxed{f = x^2 - 2xy + 2y}$$

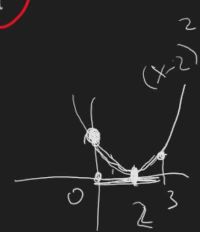
$f(0,0) = 0$, $f(3,0) = 9$

On L_1: $f = x^2$

On L_2: $\boxed{f(x,y) = 9 - 4y}$ $\quad x = 3,\ 0 \le y \le 2$.

$f(3,0) = 9$, $f(3,2) = 1$

On L_3: $f(x,2) = x^2 - 4x + 4 = (x-2)^2$



$f(2,2) = 0$, $f(0,2) = 4$

On L_4: $f(0,y) = 2y$

$f(0,0) = 0$, $f(0,2) = 4$

Step 3: The abs. max value is 9.

The abs. min value is 0. □

Lagrange Multipliers Method

Step 3: The abs. max value is 9.

The abs. min value is 0

Lagrange Multipliers Method

$f(x,y,z) \rightsquigarrow$ maximize

minimize

under some constraint

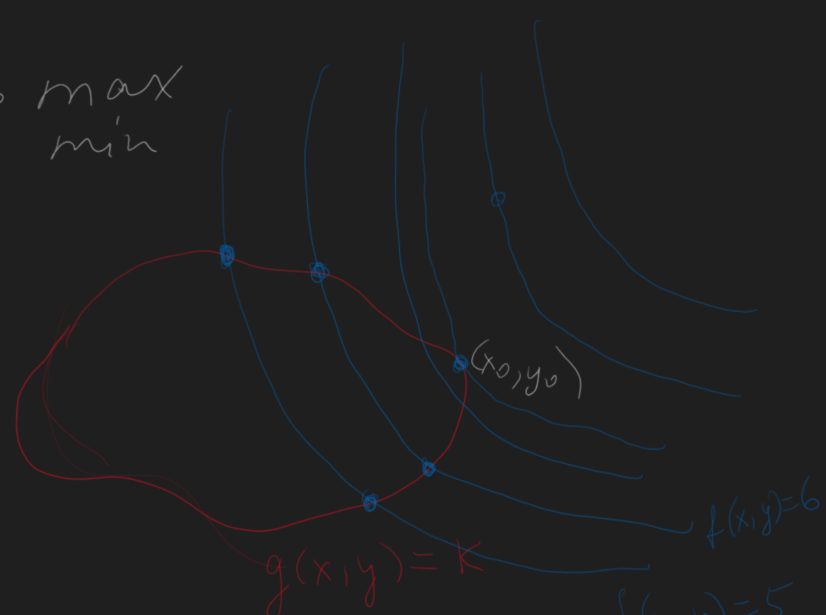
$$\boxed{g(x,y,z) = k.}$$

- Sometimes it is not possible to express z
- Even if it is possible, the formulas become too huge to deal with.

For such cases there is Lagrange Multiplier Method.

$f(x,y) \longrightarrow$ max / min

$g(x,y)=k$



Max/min at (x_0,y_0) $\Longrightarrow$ The level curve passing through (x_0,y_0) must touch the constraint curve $g(x,y)=k$.