section 14.7: 13, 15, 21, 31, 43, 45.

13: Find the local max and min values and saddle points of the function $f(x,y) = x^4 - 2x^2 + y^3 - 3y$.

Solution We find critical points:

$$\nabla f = \langle 4x^3 - 4x, 3y^2 - 3 \rangle = 0$$

$$4x^3 - 4x = 0, \quad 3y^2 - 3 = 0$$

$$x^3 - x = 0 \qquad y^2 = 1$$
$$x(x^2-1) = 0$$
$$x = 0, \pm 1 \qquad y = \pm 1.$$

6 critical points: $(0,1)$, $(0,-1)$, $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$.

Compute D and f_{xx}: $\quad D = f_{xx} f_{yy} - (f_{xy})^2$

$\boxed{f_{xx} = 12x^2 - 4}$, $f_{yy} = 6y - 3$, $f_{xy} = 0$.

$$\boxed{D = 12(3x^2-1)(2y-1)}$$

$(0,1)$: $D(0,1) < 0 \implies$ saddle point.

$(0,-1)$: $D(0,-1) > 0$, $f_{xx}(0,-1) < 0 \implies$ loc. max

$(1,1)$: $D(1,1) > 0$, $f_{xx}(1,1) > 0 \implies$ loc. min.

$(1,-1)$: $D(1,-1) < 0 \implies$ saddle point.

$(-1,1)$: $D(-1,1) > 0$, $f_{xx}(-1,1) > 0 \implies$ loc. min

$(-1,-1)$: $D(-1,-1) < 0 \implies$ saddle point.

15: Find the local max/min values and saddle points of the function

$$f(x,y) = e^x \cos y.$$

Solution: Critical points:

$$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle = 0$$

$$e^x \cos y = 0, \quad e^x \sin y = 0$$

$$\cos y = 0, \quad \sin y = 0 \quad \text{— no solutions}$$

No critical points. Hence no loc. max/min and saddle points. $\square$

21: Show that $f(x,y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

<u>Solution</u> Critical points:

$$\nabla f = \langle 2x - 4y,\ 8y - 4x \rangle = 0$$

$$2x - 4y = 0, \quad 8y - 4x = 0$$

$$x = 2y, \quad x = 2y.$$

This equation has infinitely many solutions. Namely any point of the form $(2y, y)$ is a solution. $\Longrightarrow$

$(2y, y)$ is a critical point, for any y.

Will ~~try~~ to use 2^{nd} Derivative Test to classify them.

$f_{xx} = 2$, $f_{yy} = 8$, $f_{xy} = -4$.

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 2 \cdot 8 - 4^2 = 0$$

$\Longrightarrow$ 2^{nd} Derivative Test gives no information.

Need some other way to classify our critical points. We notice:

$f(x,y) = x^2 + 4y^2 - 4xy + 12 = (x-2y)^2 + 12 \geq 12$

$f(x,y) = 12$ exactly when

$x - 2y = 0.$

$x = 2y.$

Therefore $f(x,y) = 12$ exactly at the critical points and > 12 at all other points.

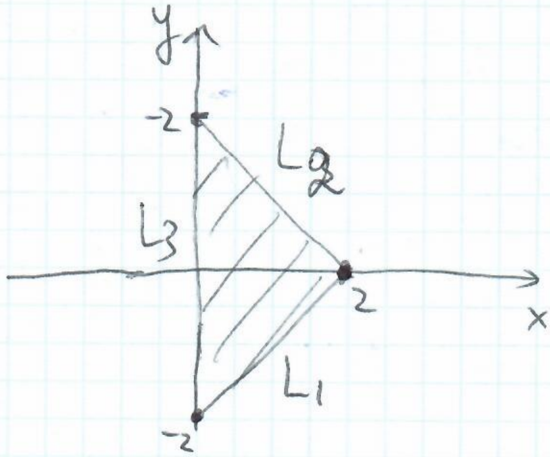
Hence all the critical points are points of absolute minimum. and the abs. min value is 12. $\square$

31: Find the absolute max and min values of f on the set D, where

$f(x,y) = x^2+y^2-2x$, D is the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$

Solution

Step 1: We find critical points.



$\nabla f = \langle 2x-2, 2y \cancel{+4} \rangle = 0$

$2x-2=0$, $2y \cancel{+4} = 0$

$x=1$, $y=0$

$(1,0)$ is a crit. point.

$\boxed{f(1,0) = }$

Step 2. We find extreme values on the boundary.

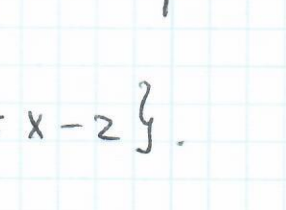
The boundary is $L_1 \cup L_2 \cup L_3$ (see the picture)

On L_1: $L_1 = \{(x,y) \mid 0 \le x \le 2, \ y = x-2\}$.

On L_1 $f(x,y) = x^2+(x-2)^2-2(x\cancel{+4}) =$

$= x^2+x^2-4x+4-2x \cancel{-4} = 2x^2-6x+4$

To find the extreme values of $2x^2-6x+4$ on $[0,2]$, let us denote $2x^2-6x+4 = g(x)$.



$g' = 4x-6 = 0$

$x = \frac{3}{2}$ – critical point.

$\boxed{g(\frac{3}{2}) = \frac{7}{2}}$, $\boxed{g(0) = 4}$, $\boxed{g(2) = 4}$

On L_2: $L_2 = \{(x,y) \mid y = 2-x, \quad 0 \le x \le 2\}$.

On L_2 $f(x,y) = x^2 + (2-x)^2 - 2x = x^2 + 4 - 4x + x^2 - 2x =$

$= 2x^2 - 6x + 4.$

Let $g(x) := 2x^2 - 6x + 4$.

$g'(x) = 4x - 6 = 0$

$x = \frac{3}{2}$ — critical point of g.

$\boxed{g\left(\frac{3}{2}\right) = \frac{7}{2}}$, $\boxed{g(0) = 4}$, $\boxed{g(2) = 4}$

On L_3: $L_3 = \{(0,y) \mid -2 \le y \le 2\}$.

On L_3 $f(x,y) = y^2$. Max value of y^2 on $[-2,2]$ is $\boxed{4}$, min is $\boxed{0}$

$f(0,\pm 2)$ $\qquad$ $f(0,0)$

Step 3 Choose smallest and biggest among the values found in steps 1 and 2!

abs. max $= 4$, abs. min $= -1$.

43. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Solution. The distance from (x, y, z) to $(4, 2, 0)$ is $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$.

For points on the cone we have $z^2 = x^2 + y^2$. Substituting it to d, we obtain $d = \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2}$

We need to find points at which d has absolute minimum. Equivalently, we need to find points at which $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2$ has absolute minimum. Therefore we consider the function $d^2 \equiv f(x, y) = (x-4)^2 + (y-2)^2 + x^2 + y^2$ defined on $\mathbb{R}^2$ and want to find points of abs. min. At first, critical points:

$$\nabla f = \langle 2(x-4) + 2x, \ 2(y-2) + 2y \rangle =$$

$$= \langle 4x - 8, \ 4y - 4 \rangle = 0.$$

$$4x - 8 = 0, \quad 4y - 4 = 0$$

$$x = 2, \quad y = 1.$$

1 critical point $(2, 1)$.

Since d^2 must have abs. min (because clearly there is a point on the cone that is closest to the point $(4, 2, 0)$) and since

abs. min can occur only at critical point,

$x=2, y=1$ is a point of abs. min. of f.

Since $z^2 = x^2 + y^2$, we have $z^2 = 5$.

$$z = \pm\sqrt{5}.$$

Hence $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$ are the points on the cone that are closest to $(4, 2, 0)$.

$\square$

Alternative solution: use Lagrange Method.

We need min of $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$.

subject to the constraint

$$g(x,y,z) = z^2 - x^2 - y^2 = 0$$

Lagrange Method applies under the assumption $\nabla g \neq 0$.

$\nabla g = \langle -2x, -2y, -2z \rangle \Rightarrow \nabla g = 0$ at $(0,0,0)$ which belongs to the cone.

Hence the points $(0,0,0)$ should be

considered separately.

$$\boxed{f(0,0,0) = 16+4+0 = 20.}$$

Now apply Lagrange Method to the case $(x,y,z) \neq (0,0,0)$.

$$\nabla f = \lambda \nabla g.$$

$$\langle 2(x-4), 2(y-2), 2z \rangle = \lambda \langle -2x, -2y, 2z \rangle.$$

$x-4 = -\lambda x$

$y-2 = -\lambda y$

$z = \lambda z$

$z^2 = x^2 + y^2$

<u>case 1</u>: $\lambda \neq 1 \implies z = 0$

$\implies x = y = 0.$

But we assumed $(x,y,z) \neq (0,0,0)$

<u>case 2</u>: $\lambda = 1 \implies x = 2$

$y = 1$

$z^2 = 5 \implies z = \pm\sqrt{5}.$

$(2,1,5)$ and $(2,1,-\sqrt{5})$.

$$\boxed{f(2,1,5) = f(2,1,-\sqrt{5}) = 10}$$ – smaller than

$\implies (2,1,\sqrt{5})$ and $(2,1,-\sqrt{5})$ are the points on the cone that are closest to $(4,2,0)$. $\square$

<u>45.</u> Find 3 positive numbers whose sum is 100 and whose product is a maximum.

<u>Solution</u> $x+y+z=100$

<u>Want</u>: xyz to be maximal possible.

We express z from the 1^{st} equation: $z=100-x-y$.

$$xyz = xy(100-x-y).$$

$$f(x,y) := xy(100-x-y) = 100xy - x^2y - xy^2.$$

We need its absolute max, which obviously must exist
(since $0 \le x \le 100$, $0 \le y \le 100$, the f-n is on the closed bounded domain)
so Extreme Value Theorem applies.

$$\nabla f = \langle 100y - 2xy - y^2,\ 100x - x^2 - 2xy \rangle = 0$$

$$y(100-2x-y)=0 \quad \text{and} \quad x(100-x-2y)=0$$

<u>case 1</u>:

$y=0,\ x=0 \implies z = 100-x-y = 100.$

$$\boxed{f(0,0)=0}$$

<u>case 2</u>: $y=0,\ 100-x-2y=0 \implies \begin{array}{l} y=0 \\ x=100 \end{array} \implies z=0$

$$\boxed{f(100,0)=0}$$

<u>case 3</u>: $100-2x-y=0,\ x=0 \implies \begin{array}{l} x=0 \\ y=100 \end{array} \implies z=0.$

$$\boxed{f(0,100)}=0$$

<u>case 4</u>: $100-2x-y=0,\ 100-x-2y=0.$

Expressing y from 1^{st} equation and substituting to 2^{nd}, we obtain

$$x=\frac{100}{3},\ y=\frac{100}{3} \implies z=\frac{100}{3}$$

$$\boxed{f\left(\frac{100}{3}, \frac{100}{3}\right) = \left(\frac{100}{3}\right)^3.}$$

On the boundary : $\{(x,y) \mid x = 0\} \cup \{(x,y) \mid y = 0\} \cup$

$$\cup \{(x,y) \mid x = 100\} \cup \{(x,y) \mid y = 100\},$$

we have :

On $\{(x,y) \mid x = 0\}$ $\quad f(0,y) = 0.$

On $\{(x,y) \mid y = 0\}$ $\quad f(x,0) = 0$

On $\{(x,y) \mid x = 100\}$ $\quad f(100,y) = -100y^2 \leq 0$

On $\{(x,y) \mid y = 100\}$ $\quad f(x,100) = -100x^2 \leq 0.$

Hence f has abs. max at $x = \frac{100}{3}$, $y = \frac{100}{3}$.

We have $z = 100 - x - y = \frac{100}{3}$. $\square$

Alternatively one can use Lagrange Method here.