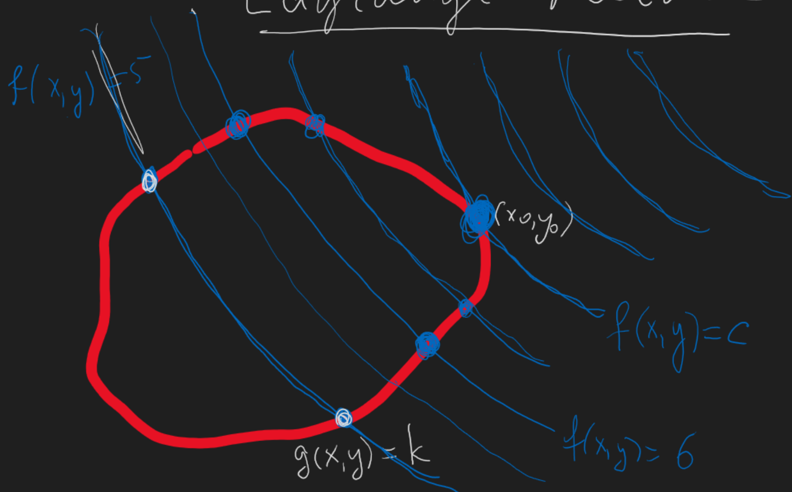
Lagrange Multipliers Method

To find max/min of

$f(x,y)$ (can be any amount of variables)

subject to the constraint $\boxed{g(x,y)=k}$

Geometrical explanation of Lagrange method

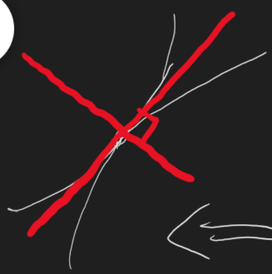


If max/min of f under the constraint $g(x,y)=k$ occurs at (x_0,y_0) $\Longrightarrow$

The level curve $\boxed{f(x,y)=c}$ passing throug (x_0,y_0) must touch the curve $\boxed{g(x,y)=k}$. $\Longleftrightarrow$

These two curves have common perpendicular line

$\Longleftrightarrow$ $\nabla f(x_0,y_0)$ is parallel to $\nabla g(x_0,y_0)$



These two curves have common perpendicular line

$\Longleftrightarrow$ $\nabla f(x_0, y_0)$ is parallel to $\nabla g(x_0, y_0)$

$$\Longleftrightarrow \boxed{\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)}$$

(assume $\nabla g(x_0, y_0) \neq 0$).

Lagrange Multiplier Method.

To find max/min of f subject to the constraint $g(x,y) = k$

[assuming these max/min exist and $\nabla g \neq 0$ at points satisfying $g(x,y)=k$]

(a) Find all points x, y and λ such that

$\nabla f(x,y) = \lambda \nabla g(x,y)$. and

$g(x,y) = k$

(b) Evaluate f at pts obtained in (a) and choose the biggest and the smallest.

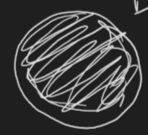
Ex. Find the extreme values of

$f(x,y) = xy$ on the disc

Ex. Find the extreme values of $f(x,y) = x\cdot y$ on the disc

$$\boxed{x^2+y^2 \le 1}.$$

D



Solution

We have such strategy:

1 step Find critical pts and values of f at them.

2 step Find extreme values at the boundary

Often one can do step 2 using Lagrange Method.

3 step Among the values obtained in 1^{st} and 2^{nd} steps choose the biggest and the smallest.

1 step. $\nabla f = 0$

$$\nabla f = \langle y, x \rangle = 0$$

$$x = 0, \ y = 0.$$

1 crit. pt $(0,0)$. $f(0,0) = 0$

2 step We need to find extreme values of f at the boundary

$$g(x,y) = \boxed{x^2+y^2 = 1}$$

Will apply Lagrange Method.

[max/min $\exists$, since the circle is a bounded and closed set.

$\boxed{\nabla g = \langle 2x, 2y \rangle}$ $\neq 0$ on the circle.

Hence Lagrange Method applies].

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$\nabla f = \lambda \nabla g$, $x^2+y^2=1$.

$\nabla f = \langle y, x \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\langle y, x\rangle = \cancel{\lambda} \langle 2x, 2y \rangle$, $x^2+y^2=1$.

$$\begin{aligned} y &= \lambda \cdot 2x \\ x &= \lambda \cdot 2y \\ x^2+y^2 &= 1 \end{aligned} \Longrightarrow \begin{aligned} y &= \lambda \cdot 2x \\ x &= \lambda \cdot 2 \cdot \lambda \cdot 2x \\ x^2+y^2 &= 1 \end{aligned} \Longrightarrow$$

$$\Longrightarrow \boxed{\begin{aligned} &\boxed{y = \lambda \cdot 2x} \\ &(1-4\lambda^2)x = 0 \\ &\boxed{x^2+y^2=1} \end{aligned}} \Longrightarrow$$

2 cases: $x=0$ or $1-4\lambda^2=0$.

1 case: $x=0$ — not possible.

$y=0$

Then $x^2+y^2 \neq 1$.

2 case: $1-4\lambda^2=0$, $4\lambda^2=1$, $\lambda = \pm\frac{1}{2}$.

$\lambda=\frac{1}{2}$: $\begin{aligned} y&=x \\ 2x^2&=1 \end{aligned} \Rightarrow y=x=\pm\frac{1}{\sqrt{2}}$.

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$, $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$$\underline{\lambda = -\frac{1}{2}}: \quad \begin{array}{l} y = -x \\ 2x^2 = 1 \end{array} \implies \begin{array}{l} x = \pm\frac{1}{\sqrt{2}} \\ y = -x. \end{array}$$

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}, \quad f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$$

Extreme values on the boundary are $\boxed{\frac{1}{2}}$ and $\boxed{-\frac{1}{2}}$.

<u>3 step</u> choose the biggest and the smallest among the values obtained in steps 1 and 2:

$\frac{1}{2}$ – max

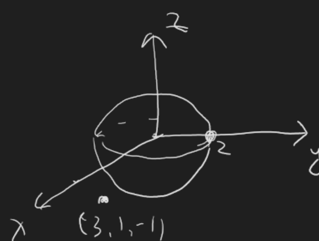
$-\frac{1}{2}$ – min. □

λ is called Lagrange Multiplier.

<u>Ex.</u> Find the pts on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$.

Solution

Ex. Find the pts on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$.



Solution

Want to find min/max of the distance from (x,y,z) to $(3,1,-1)$

$$d(x,y,z) = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$$

subject to the constraint

$$g(x,y) = x^2+y^2+z^2 = 4$$

Instead of $d(x,y,z)$ we can find max/min of $\boxed{f(x,y,z) = (x-3)^2+(y-1)^2+(z+1)^2}$

(just to make formulas easier)

Want to find max/min of f

subject to the constraint

$$g(x,y) = x^2+y^2+z^2 = 4.$$

[Max/min $\exists$, since the sphere is a closed bounded set.

$\nabla g = \langle 2x, 2y, 2z \rangle \neq 0$ at points on the sphere.

Hence Lagrange Method applies].

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$$\nabla f = \lambda \nabla g$$
$$x^2+y^2+z^2=4$$

$$\nabla f = \langle 2(x-3),\ 2(y-1),\ 2(z+1)\rangle$$
$$\nabla g = \langle 2x, 2y, 2z\rangle$$

$$\begin{cases} \cancel{2}(x-3) = \lambda\cdot \cancel{2}x \\ 2(y-1) = \lambda\cdot 2y \\ 2(z+1) = \lambda\cdot 2z \\ x^2+y^2+z^2=4 \end{cases} \iff \begin{cases} (1-\lambda)x = 3 \\ (1-\lambda)y = 1 \\ (1-\lambda)z = -1 \\ x^2+y^2+z^2=4 \end{cases} \iff$$

$$x = \frac{3}{1-\lambda},\quad y = \frac{1}{1-\lambda},\quad z = -\frac{1}{1-\lambda} \iff$$

$$\frac{3^2+1^2+(-1)^2}{(1-\lambda)^2} = 4$$

$$\frac{11}{(1-\lambda)^2} = 4 \qquad (1-\lambda)^2 = \frac{11}{4} \qquad (1-\lambda) = \pm\frac{\sqrt{11}}{2}$$

$\underline{1-\lambda = \frac{\sqrt{11}}{2}}$: $x = \frac{6}{\sqrt{11}},\ y = \frac{2}{\sqrt{11}},\ z = -\frac{2}{\sqrt{11}}$

$\boxed{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)}$ - closest (min)

$\underline{1-\lambda = -\frac{\sqrt{11}}{2}}$: $x = -\frac{6}{\sqrt{11}},\ y = -\frac{2}{\sqrt{11}},\ z = \frac{2}{\sqrt{11}}$

$\boxed{\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}$

farthest

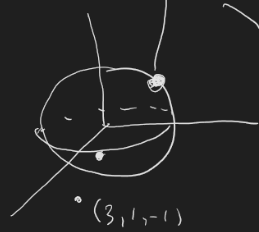
$1-\lambda = \frac{\sqrt{11}}{2}$: $x = \frac{6}{\sqrt{11}}$, $y = \frac{2}{\sqrt{11}}$, $z = -\frac{2}{\sqrt{11}}$ (min)

$\boxed{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)}$ – closest

$1-\lambda = -\frac{\sqrt{11}}{2}$: $x = -\frac{6}{\sqrt{11}}$, $y = -\frac{2}{\sqrt{11}}$, $z = \frac{2}{\sqrt{11}}$

$\boxed{\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}$.

farthest (max).



□

Kinds of problems on topics of Chapter 14 which you need to know how to solve for exam:

- find ∇f
- find $D_{\bar{u}} f$
- find and classify critical pts
- find extreme values of a f-n on a closed bounded set. (often for that you need Lagrange Methods).

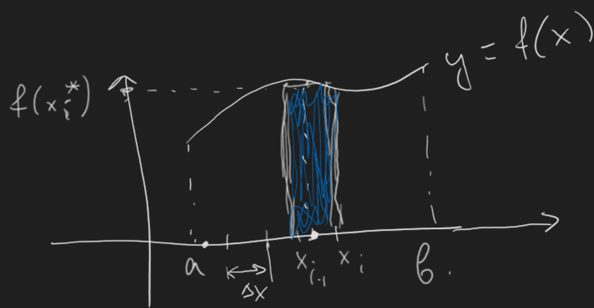
Double integrals.

Double integrals.

About integrating f-s of several variables.

Review of definite integrals of f-s of 1 variable.

$\int_a^b f(x)\,dx$



Divide $[a, b]$ into n intervals $[x_{i-1}, x_i]$

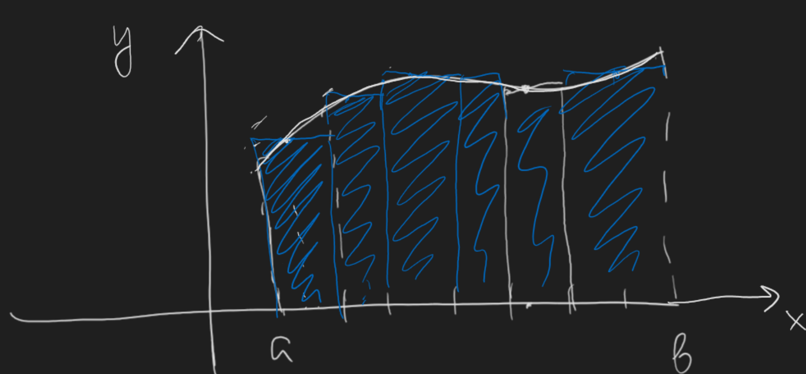
of length $\Delta x = \frac{b-a}{n}$

Choose a pt x_i^* in $[x_{i-1}, x_i]$.

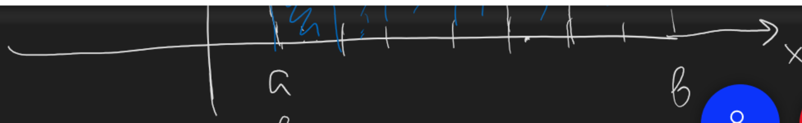
Def.

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} \boxed{f(x_i^*)\,\Delta x}$$

area of blue rectangle.

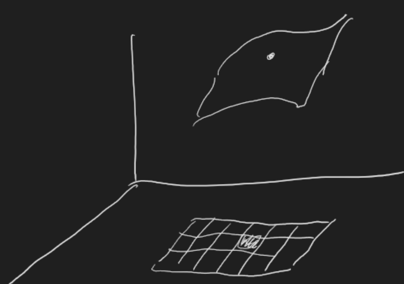


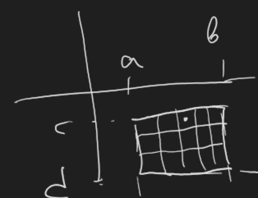
Geometrical sense of $\int_a^b f(x)\,dx$: it is the area under the graph of f from a to b.



Geometrical sense of $\int_a^b f(x)\,dx$: it is the area under the graph of f from a to b. (if f is a positive f-n).

For f-s of 2 variables:





Divide the rectangle into small subrectangles, choose a pt at each of them, (x_i^*, y_i^*).

$$\sum f(x_i^*, y_i^*) \underbrace{\Delta A}_{\text{area of subrectangle.}}$$