Previous lecture:

- How to find extreme values of a continuous f-n on a closed bounded set.
  - (a) find critical pts and the value of f at them.
  - (b) find extreme values at the boundary
  - (c) Choose the biggest and the smallest among the values obtained in (a) and (b).

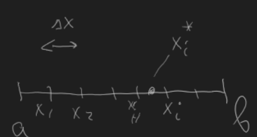
- How to find max/min of $f(x,y,z)$ subject to a constraint $g(x,y,z) = k$.

Lagrange Method: if f has min/max at (x_0, y_0, z_0) then

$$\boxed{\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)}$$

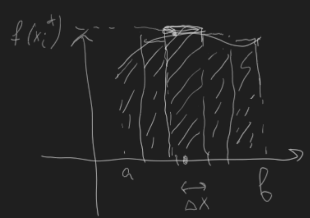
$$\boxed{g(x_0,y_0,z_0) = k.}$$

- $\int_a^b f(x)\,dx$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

area under the graph of f from a to b.



Double integrals over rectangles.

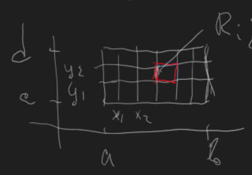
$f(x,y)$ on $R = [a,b] \times [c,d] =$

d + R_{ij} $= \{(x,y) \mid a \le x \le b,\ c \le y \le d\}.$

Double integrals over rectangles

$f(x,y)$ on $R = [a,b] \times [c,d] =$

$= \{(x,y) \mid a \le x \le b,\ c \le y \le d\}.$



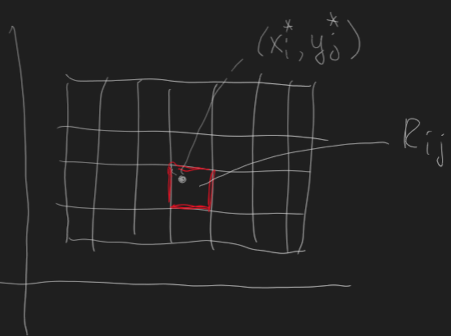
Divide $[a,b]$ into n subintervals of length

$$\Delta x = \frac{b-a}{n}.$$

Divide $[c,d]$ into m subintervals of length

$$\Delta y = \frac{d-c}{m}.$$

Then R is divided into subrectangles, R_{ij} of area $\Delta A = \Delta x \cdot \Delta y$



Def.

$$\iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{f(x_i^*, y_j^*)\,\Delta A}$$

if this limit exists.

Remark This lim exists for all contin. f-s and for many other f-s.

Remark There is also notation

$$\iint_R f(x,y)\,dA \equiv \iint_R f(x,y)\,dx\,dy$$

Geometrical sense of $\iint_R f(x,y)\,dx\,dy$

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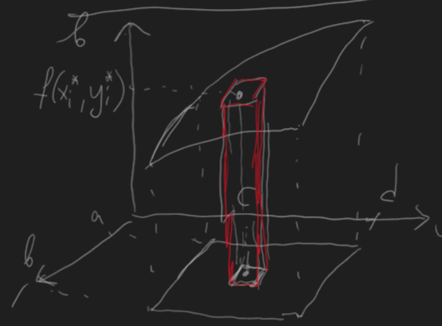


$\boxed{f(x_i^*, y_j^*)\cdot \Delta A} \approx$ volume of the red column.

$$\sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\cdot \Delta A \;\boxed{\approx}$$

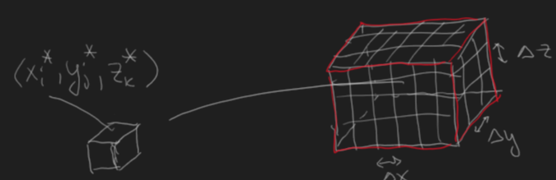
the volume below the graph above R.

$$\iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\cdot \Delta A \;\boxed{=}$$

the volume below the graph above R.

Integrals of f-s of 3 or more variables.

$f(x,y,z)$ on $I=[a,b]\times[c,d]\times[e,f]$



$\Delta V = \Delta x\cdot\Delta y\cdot\Delta z$

$$\lim_{\substack{i\to\infty \\ j\to\infty \\ k\to\infty}} \sum_k \sum_j \sum_i f(x_i^*, y_j^*, z_k^*)\cdot \Delta V$$

Def. The triple integral of $f(x,y,z)$ on the rectangular box I is

$$\iiint_I f(x,y,z)\,dV \equiv \iiint_I f(x,y,z)\,dx\,dy\,dz := $$

$$\lim_{\substack{i\to\infty \\ j\to\infty \\ k\to\infty}} \sum_i \sum_j \sum_k f(x_i^*, y_j^*, z_k^*)\cdot \Delta V$$

How to compute double integrals?

Microsoft Whiteboard

How to compute double integrals

For f-s of 1 variable we use Fundamental Theorem of Calculus to compute $\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \underbrace{F(b)}_{\text{antiderivative of } f} - F(a).$$

Computing double integrals we will reduce to the case of f-s of 1 variable,

using so called iterated integrals.

Iterated Integrals.

$f(x,y)$ on $R = [a,b] \times [c,d]$.

Assume y is const. $\int_a^b f(x,y)\,dx$ — a f-n of y

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \equiv \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

- iterated integral.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \equiv \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Fubini's Theorem If f is a continuous f-n on rectangle $R = [a,b] \times [c,d]$, then

Fubini's Theorem

If f is a continuous f-n on rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Idea of the proof

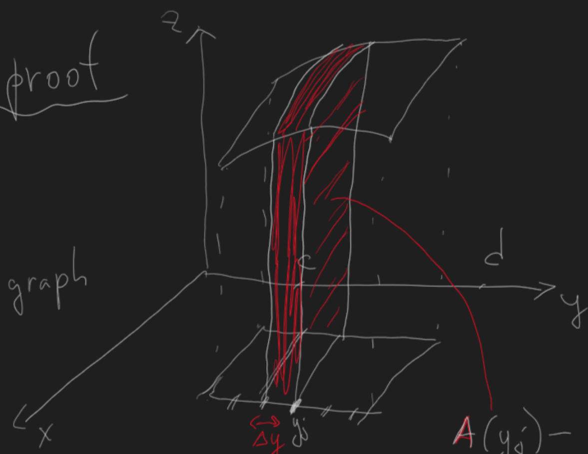
$\boxed{\iint_R f(x,y)\,dA} =$

= volume under the graph

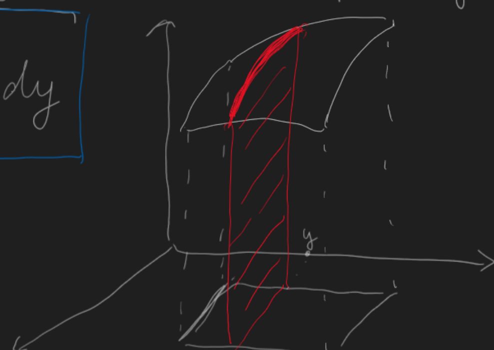
$\approx \sum$ of volumes of the layers of width Δy =

$= \sum_j A(y_j)\cdot \Delta y \approx \int_c^d A(y)\,dy =$

$= \boxed{\int_c^d \int_a^b f(x,y)\,dx\,dy}$



$A(y_j)$ – area of the cross-section by the plane $y = y_j$

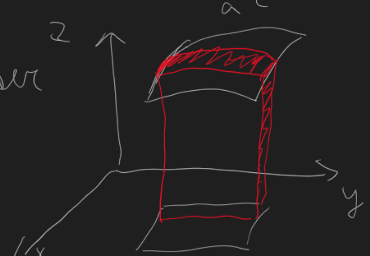


$A(y)$ is the area under the graph of $f(x,y)$ where y is fixed.

$A(y) = \int_a^b f(x,y)\,dx$

To show that $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$

one should consider



$\square$

Remark Fubini's Th. holds for f-s of 3, 4, ... variables.

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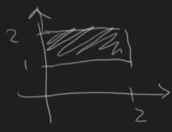
Ex. Find $\iint_R (x-3y^2)\,dA$, where $R=[0,2]\times[1,2]$

Solution



$$\iint_R (x-3y^2)\,dA = \int_0^2 \boxed{\int_1^2 (x-3y^2)\,dy}\,dx =$$

$$= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx = \int_0^2 \left((2x-8)-(x-1)\right) dx =$$

$$= \int_0^2 (x-7)\,dx = \left[\frac{x^2}{2} - 7x\right]_{x=0}^{x=2} = \frac{4}{2} - 7\cdot 2 = -12.$$

2nd Solution $\iint_R (x-3y^2)\,dA = \int_1^2 \boxed{\int_0^2 (x-3y^2)\,dx}\,dy =$

$$= \int_1^2 \left[\frac{x^2}{2} - 3y^2 x\right]_{x=0}^{x=2} dy = \int_1^2 (2-6y^2)\cdot dy =$$

$$= \left[2y - 2y^3\right]_{y=1}^{y=2} = (4-2\cdot 8) - (2-2) = -12.$$

Ex. $\iiint_{[0,1]^3} xz e^{xy}\,dV = \int_0^1\int_0^1 \boxed{\int_0^1 xz e^{xy}\,dy}\,dx\,dz$

$$= \int_0^1\int_0^1 \left[\cancel{x} z \frac{e^{xy}}{\cancel{x}}\right]_{y=0}^{y=1} dx\,dz = \int_0^1 \boxed{\int_0^1 (z e^{x} - z)\,dx}\,dz =$$

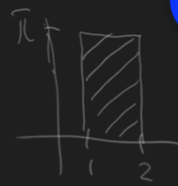
$$= \int_0^1 \left[z e^{x} - z x\right]_{x=0}^{x=1} dz = \int_0^1 \left((ze - z) - z\right) dz =$$

$$= \int_0^1 (ze - 2z)\,dz = \int_0^1 z(e-2)\,dz = \left[(e-2)\frac{z^2}{2}\right]_{z=0}^{z=1} =$$

$$= \frac{e-2}{2}$$ □

Ex. $\iint y \sin(xy)\,dA =$

Ex. $\iint\limits_{[1,2]\times[0,\pi]} y \sin(xy)\, dA =$



$=$

Poll: what is easier

(a) integrate at first over x

(b) —//— over y

$$= \int_0^{\pi} \boxed{\int_1^2 y \sin(xy)\, dx}\, dy = \int_0^{\pi} \left[-\cancel{y}\,\frac{\cos(xy)}{\cancel{y}}\right]_{x=1}^{x=2} dy =$$

$$= \int_0^{\pi} \left(-\cos(2y) + \cos y\right) dy = \left[-\frac{\sin(2y)}{2} + \sin y\right]_{y=0}^{y=\pi} = 0$$

□