14.8.9 find extremum value of

$$\begin{cases} f(x,y,z) = xy^2z \\ x^2+y^2+z^2 = 4 \end{cases}$$

$$\begin{cases} f(x,y,z) \\ g(x,y,z) = k \end{cases}$$

<u>recall</u> <u>Lagrange methode</u>

Assume extremum exists and $\nabla g \neq 0$ on $g(x,y,z)=k$

a) find (x,y,z) such that it exists $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$$

b) evaluate f on those points and take extremum one

So we look for (x,y,z) and λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$$\begin{cases} \partial_x f(x,y,z) = \lambda \partial_x g(x,y,z) \\ \partial_y f = \lambda \partial_y g \\ \partial_z f = \lambda \partial_z g \end{cases}$$

⚠ We check that $\nabla g \neq 0$

$\nabla g(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \neq 0$ on $g(x,y,z)=4$

since $g=0 \Rightarrow (x,y,z) = (0,0,0)$

and $(0,0,0) \notin g(x,y,z)=4$

$$\Rightarrow \begin{cases} y^2 z = \lambda 2x \\ 2xyz = \lambda 2y \\ xy^2 = \lambda 2z \end{cases}$$

we solve the previous equation and $x^2+y^2+z^2=4$

1) we see $f(x,y,z) = 0$ if $x=0$ or $y=0$ or $z=0$

2) from $2xyz = \lambda 2y$ we have $2(xz-\lambda)y = 0$

$\boxed{\bullet\ y = 0}$ or $\bullet\ xz - \lambda = 0$

if $xz - \lambda = 0$ we have

$$\begin{cases} y^2 z = 2x^2 z \\ 2xyz = 2xyz \\ xy^2 = 2xz^2 \\ x^2+y^2+z^2 = 4 \end{cases} \Rightarrow \begin{cases} z(y^2 - 2x^2) = 0 \\ x(y^2 - 2z^2) = 0 \\ x^2+y^2+z^2 = 4 \end{cases}$$

if $\bullet$ $\boxed{z = 0 \text{ or } x = 0}$

$\Rightarrow$ if $\bullet$ $\boxed{z \neq 0 \text{ and } x \neq 0}$

$$\begin{cases} y = \pm x\sqrt{2} \\ y = \pm z\sqrt{2} \\ \frac{y^2}{2} + y^2 + \frac{y^2}{2} = 4 \end{cases} \Rightarrow \boxed{\begin{array}{l} x = \pm 1 \\ y = \pm\sqrt{2} \\ z = \pm 1 \end{array}}$$

because $y^2 z = \lambda 2x$ and $\lambda = xz$ so we replace

- there are several cases but each one of them with $x=0$ or $y=0$ or $z=0$ leads to $f(x,y,z)=0$

So we look at other points and if f is bigger or smaller on those points then there is no need to compute the $x=0$ or $y=0$ or $z=0$

So we evaluate f at $x=\pm 1$, $y=\pm\sqrt{2}$, $z=\pm 1$

f has symetries so $f(1,\pm\sqrt{2},1) = f(-1,\pm\sqrt{2},-1) = 1\times(\pm\sqrt{2})^2\times 1 = 2$

$$f(-1,\pm\sqrt{2},1) = f(1,\pm\sqrt{2},-1) = (-1)\times(\pm\sqrt{2})^2\times 1 = -2$$

Conclusion

- maximum of f is 2 and it is reached at $\left\{ (1,\sqrt{2},1),(1,-\sqrt{2},1),(-1,\sqrt{2},-1), (-1,-\sqrt{2},-1) \right\}$
- minimum of f is -2 reached at $\left\{ (-1,\sqrt{2},1),(-1,-\sqrt{2},1),(1,\sqrt{2},-1),(1,-\sqrt{2},-1) \right\}$

14.8.21 find extremum of $f(x,y) = x^2+y^2+4x-4y$ on $D = x^2+y^2 \le 9$

recall thm 14.7.8

f ctn on a closed bounded set D then f attains an absolute max and min

Strategy: a) find the extremum on the interior of D at critical point

b) " " " " " border of D

c) take the maximum and minimum

— f is continuous on D (polynomial function)

— D is closed and bounded

↳ thm 14.7.8 applies and we follow the strategy.

a) $\mathring{D}$ = interior of $D = \{(x,y) \in \mathbb{R}^2, \; x^2+y^2 < 9\}$

critical point ($\nabla f = 0$)

$$\begin{cases} \partial_x f(x,y) = 2x+4 = 0 \\ \partial_y f(x,y) = 2y-4 = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = 2 \end{cases}$$ and $(-2)^2+2^2 = 8 < 9$ so it belongs to $\mathring{D}$

and $f(-2,2) = 4+4-8-8 = -8$

b) with $g(x,y) = x^2+y^2$ we have $\nabla g(x,y) = (2x,2y) \neq (0,0)$ on

∂D = border of $D = \{(x,y) \in \mathbb{R}^2, \; x^2+y^2 = 9\}$

($\nabla g(x,y) = 0 \Rightarrow (2x,2y) = (0,0) \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$ but $g(0,0) = 0 \neq 9$

$\Rightarrow$ we can apply Lagrange method we look $(x,y)\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$ such that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 9 \end{cases} \Rightarrow \begin{cases} \partial_x f = \lambda \partial_x g \\ \partial_y f = \lambda \partial_y g \\ g(x,y) = 9 \end{cases} \Rightarrow \begin{cases} 4 = \lambda 2x \\ -4 = \lambda 2y \\ x^2+y^2 = 9 \end{cases}$$

rk: on ∂D we have $x^2+y^2=9$ so $f(x,y) = x^2+y^2+4(x-y) = 9+4(x-y)$

$\lambda \neq 0$ otherwise $4=0$ so $\begin{cases} x = 2/\lambda \\ y = -2/\lambda \\ 8 = 9\lambda^2 \end{cases} \Rightarrow \begin{cases} \lambda = \pm \frac{2\sqrt{2}}{3} \\ x = \pm 3/\sqrt{2} \\ y = \mp 3/\sqrt{2} \end{cases}$

$$f\left(\pm\frac{3\sqrt{2}}{2}, \pm\frac{3\sqrt{2}}{2}\right) = 9+4\times\frac{3\sqrt{2}}{2}(\pm 1 - \pm 1) = 9+6\sqrt{2}(\pm 1 \mp 1)$$

$f\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = f\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = 9$ and $f\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = 9-12\sqrt{2}$

$$f\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = 9 + 6\sqrt{2}\,(1-(-1)) = 9 + 12\sqrt{2}$$

Conclusion

- maximum is $9 + 12\sqrt{2}$ reached on $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) \in \partial D$

- minimum is -8 (since $9 - 12\sqrt{2} > -8$) reached on $(-2, 2) \in \mathring{D}$

on ∂D we have to solve $\begin{cases} \max f(x,y) \\ g(x,y) = 9 \end{cases}$

14.8.25 find extremum of $f(x,y) = x$ on $D = \{(x,y),\ y^2 + x^4 - x^3 = 0\}$

a) We try with Lagrange method

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 1 = \lambda(4x^3 - 3x^2) \\ 0 = 2y\lambda \\ y^2 + x^4 - x^3 = 0 \end{cases}$$

$\lambda \neq 0$ since $1 \neq 0$

So $y = 0$ and then $\begin{cases} x^3(x-1) = 0 \\ 1 = \lambda x^2(4x-3) \end{cases}$

$x \neq 0$ since $1 \neq 0$

So $x = 1$ and $\lambda = 1$

so extremum of f is $f(1,0)=1$ reached on $(1,0)$

b) Show that $f(0,0)=0$ is the minimum

rk : $y^2+x^4-x^3=0 \Rightarrow y^2+x^4=x^3 \geqslant 0$

$$\Rightarrow x \geqslant 0$$

so $f(x,y)=x \geqslant 0$

since $f(0,0)=0$ and $(0,0) \in \{(x,y),\ y^2+x^4-x^3=0\}$

0 is the minimum of f

$\rightarrow$ contradiction with $f(1,0)=1$ is an extremum

c) We haven't checked that $\nabla g \neq 0$ on $g(x,y) = 0$

$$\nabla g(x,y) = (4x^3 - 3x^2, 2y)$$

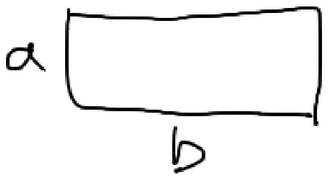
$$(0,0) \in \{(x,y), g(x,y) = 0\}$$

$$\nabla g(0,0) = 0$$

We can not apply Lagrange method.

14.8.24

Show that the rectangle with the maximum area with fix perimeter p is a square



Area $= ab$

perimeter $= 2(a+b)$

solve $\begin{cases} \max ab \\ 2(a+b) = p \end{cases}$

$\nabla g = (1,1) \neq 0$ we apply Lagrange method

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(a,b) = p \end{cases} \Rightarrow \begin{cases} \partial_a f = b = \lambda 2 \\ \partial_b f = a = \lambda 2 \\ 2(a+b) = p \end{cases} \Rightarrow \begin{cases} a = b = 2\lambda \\ \lambda = \frac{p}{4} \end{cases} \Rightarrow \begin{cases} a = p/2 \\ b = p/2 \\ \lambda = p/4 \end{cases}$$

max Area $= p^2/4$ reached on $(p/2, p/2)$ which is a square.

14.8.23 find extremum of $f(x,y) = e^{-x^2y}$ on $x^2+4y^2 \leq 1$

- f is continuous on D
- D is bounded and closed

on $\mathring{D} = \{(x,y) \in \mathbb{R}^2, \ x^2+4y^2 < 1\}$

- critical point $(\nabla f = 0)$ $\begin{cases} -y \, e^{-xy} = 0 \\ -x \, e^{-xy} = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

we check that $(0,0) \in \mathring{D}$ since $0^2 + 4 \cdot 0^2 < \underline{1}$

$f(0,0) = \underline{1}$

— on $\partial D = \{ (x,y) \in \mathbb{R}^2, \ x^2+4y^2 = 1 \}$

We check that $\nabla g \neq 0$ on ∂D

$\nabla g(x,y) = (2x, 8y)$ and $\nabla g = 0 \Rightarrow (x,y) = (0,0)$

but $(0,0) \notin \partial D$ since $0^2 + 4\,0^2 \neq 1$

So $\nabla g \neq 0$ on ∂D

We can apply Lagrange method

We look $(x,y) \in \mathbb{R}^2, \lambda \in \mathbb{R}$ st $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases}$

$\Rightarrow \begin{cases} -y e^{-xy} = \lambda 2x & \times x \\ -x e^{-xy} = \lambda 2y & \times y \\ x^2+4y^2 = 1 \end{cases}$ we have $\lambda 2x^2 = \lambda 2 y^2 \Rightarrow 2\lambda(x^2-y^2) = 0$

$\lambda \neq 0$ otherwise $(x,y) = (0,0) \notin \partial D$

so $x^2 = y^2 \Rightarrow x = \pm y$

so $x^2 + 4x^2 = 1$ so $x = \pm \frac{\sqrt{5}}{5}$

$$f\left(\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right) = f\left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right) = e^{1/5}$$

$$f\left(-\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right) = f\left(\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right) = e^{-1/5}$$

Conclusion since $e^{-1/5} < 1 < e^{1/5}$

max is $e^{1/5}$ reached on $\left(\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$ and $\left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$

min is $e^{-1/5}$ reached on $\left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ and $\left(-\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$