## Previous lecture:

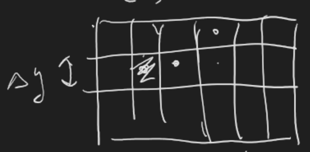
defined $\iint_R f(x,y)\,dA :=$

$R$ – rectangle

$$= \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

$\Delta x$

$\Delta y$



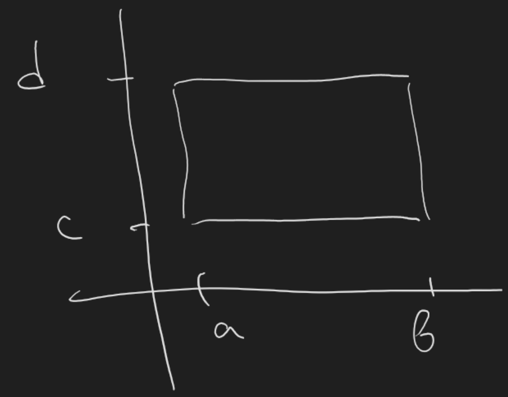
$\Delta A = \Delta x \Delta y$

Geom. sense: If $f \geq 0$, then

$$\iint_R f(x,y)\,dA = \text{volume under the graph of } f.$$

## Fubini's Th.

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$



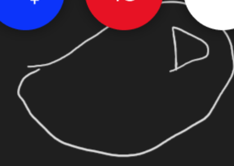
## Double integrals over general regions.

$f(x,y)$ on $D$ – bounded region

How to define $\iint_D f(x,y)\,dA$?

# Double integrals over general regions.

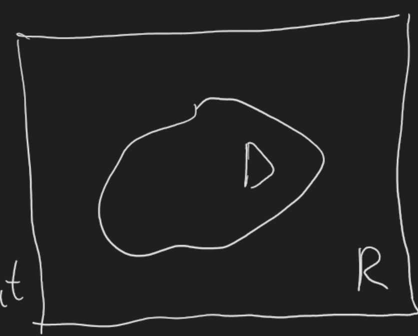
$f(x,y)$ on $D$ – bounded region




How to define $\iint_D f(x,y)\,dA$?

Will reduce to the case of rectangles.

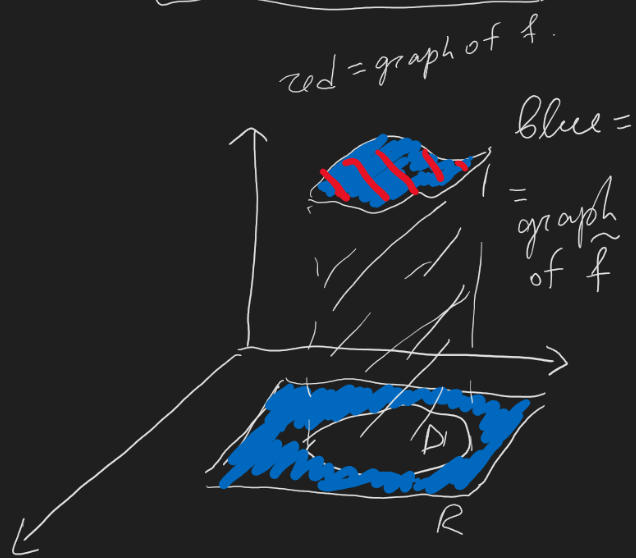
Consider




$$\tilde f(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

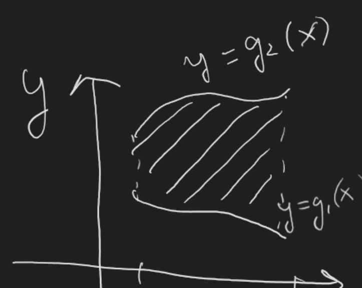
<u>Def.</u> $\iint_D f(x,y)\,dA := \boxed{\iint_R \tilde f(x,y)\,dA}$.

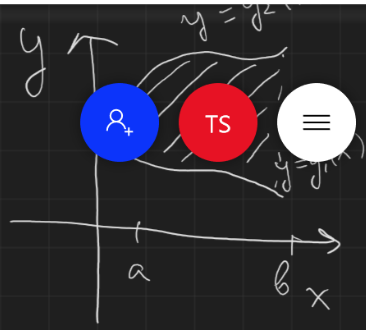
<u>Geom sense:</u> Suppose $f \geq 0$.




$\iint_D f(x,y)\,dA$ still is interpreted as the volume under the graph of $f(x,y)$.

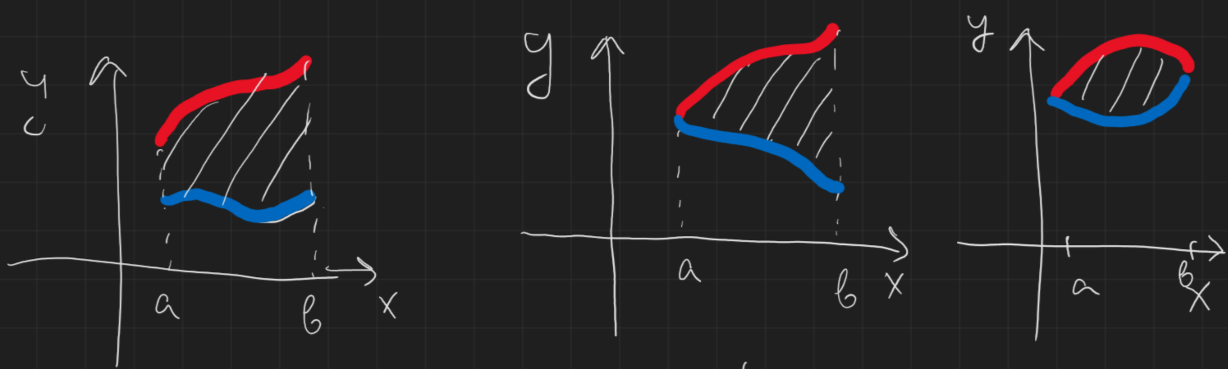
<u>Def.</u> We say that $D$ is of type I if



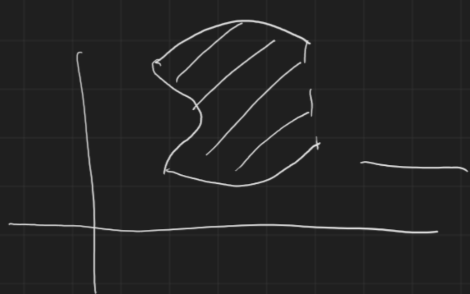



Def. We say that D is of type I if it lies between the graphs of 2 f-s of x, that is

$$D=\{(x,y) \mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}.$$

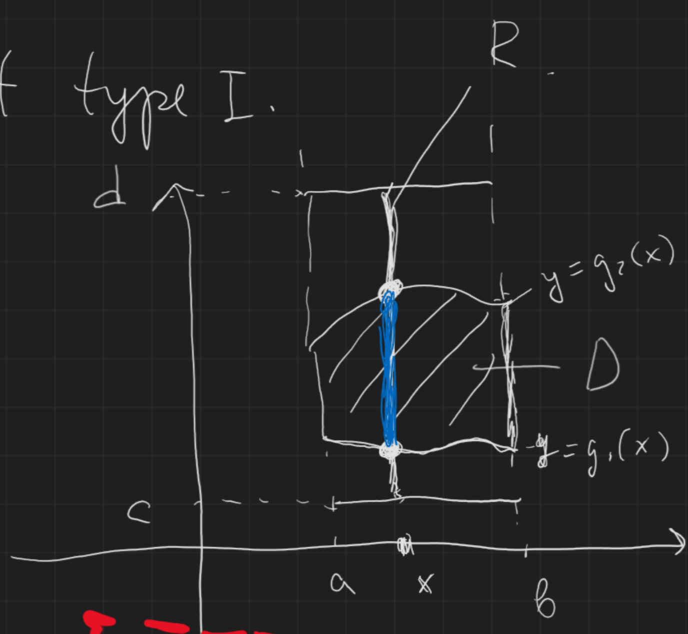



type I

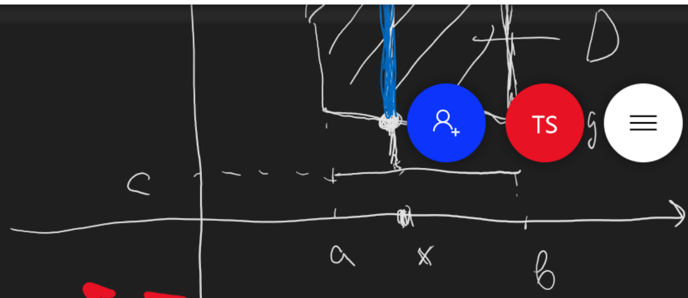


not type I.

Suppose D is of type I.




$$\iint_D f(x,y)\,dA \overset{\text{def.}}{=}$$




$$\iint_D f(x,y)\,dA \overset{def.}{=}$$

$$= \iint_R \tilde{f}(x,y)\,dA \overset{\text{Fubini}}{=} \int_a^b \boxed{\int_c^d \tilde{f}(x,y)\,dy}\,dx =$$

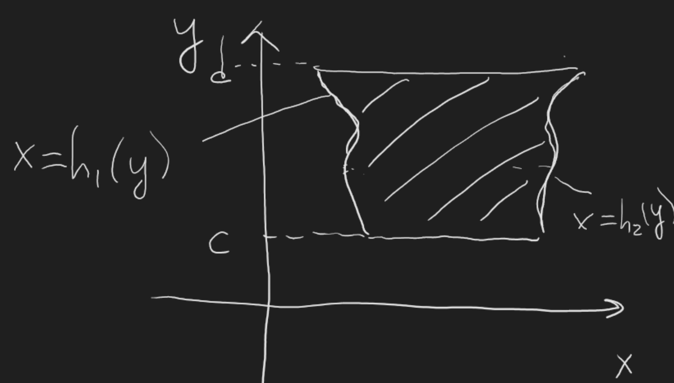
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

We obtained

If $D$ is of type I

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

then $$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\underline{\underline{dy}}\,dx$$

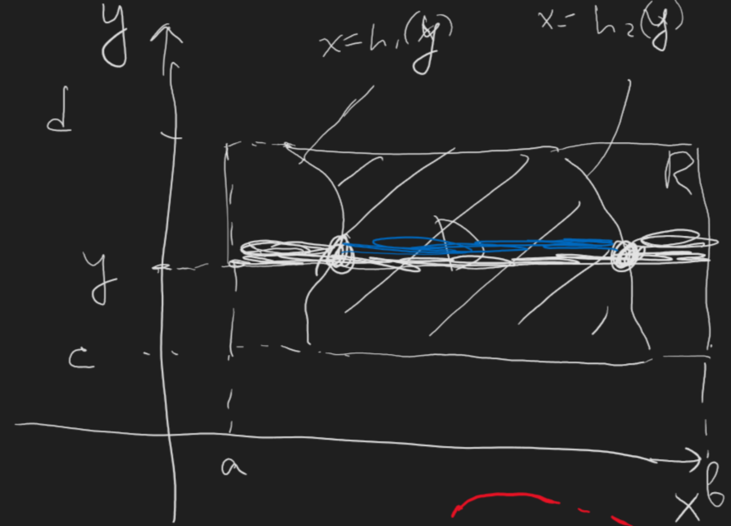



<u>Def.</u> We say $D$ is of type II if it lies between the graphs of 2 f-s of $y$, that is

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

Def. We say $D$ is of type II if it lies between the graphs f-s of $y$, that is

$$D=\{(x,y)\mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$$




$$\iint_D f(x,y)\,dA \overset{\text{def}}{=} \iint_R \tilde{f}(x,y)\,dA \overset{\text{Fubini}}{=} \int_c^d \boxed{\int_a^b \tilde{f}(x,y)\,dx}\,dy$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

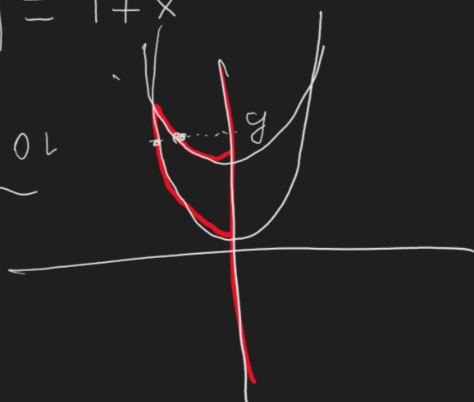
So we obtained

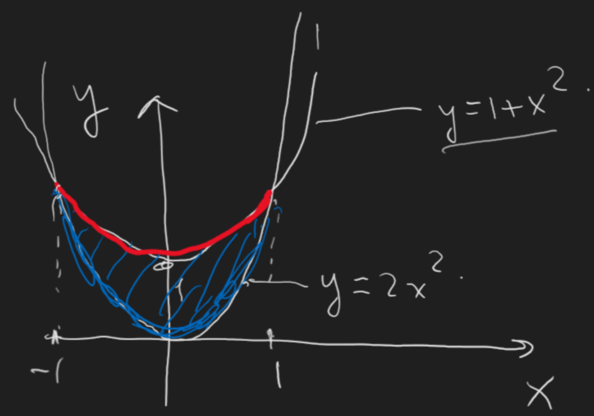
$$\boxed{\begin{array}{l}\text{If } D \text{ is of type II} \\ D=\{(x,y)\mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}, \\ \text{then } \displaystyle\iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.\end{array}}$$

Ex. Find $\iint y\,dA$, where $D$ is the

Ex. Find $\iint_D y\,dA$, where D is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$

Solution




Poll: Is D of type I or type II?

(a) I

(b) II.

D is type I

Points of intersection of those parabolas:

$2x^2=1+x^2$

$x^2=1, \quad x=\pm 1.$

$$D=\{(x,y)\mid -1\le x\le 1,\ 2x^2\le y\le 1+x^2\}.$$

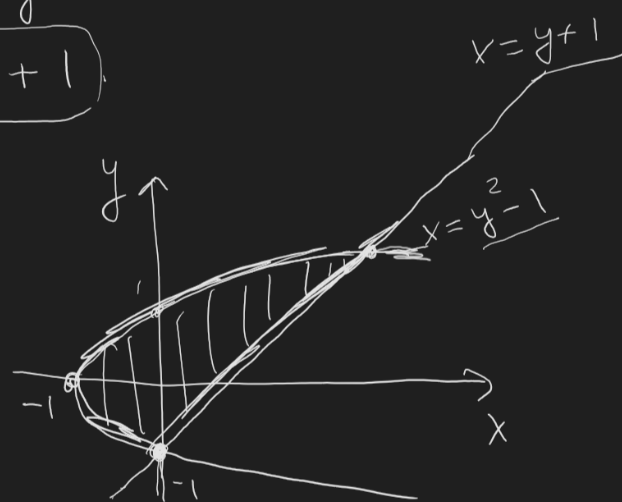
$$\iint_D y\,dA=\int_{-1}^{1}\int_{2x^2}^{1+x^2} y\,dy\,dx=\int_{-1}^{1}\left[\frac{y^2}{2}\right]_{y=2x^2}^{y=1+x^2}dx=$$

$$=\int_{-1}^{1}\left(\frac{(1+x^2)^2}{2}-\frac{(2x^2)^2}{2}\right)dx=\int_{-1}^{1}\frac{1}{2}\left(1+2x^2+x^4-4x^4\right)dx=$$

$$= \int_{-1}^{1} \left( \frac{(1+x^2)^2}{2} - \frac{(2x^2)^2}{2} \right) dx = \int_{-1}^{1} \frac{1}{2}\left(1+2x^2 \ldots\right) d\ldots$$

$= \ldots$ (easy). □

<u>Ex.</u> Find $\iint_D xy \, dA$, where $D$ is the region bounded by the line $y = x-1$ and the parabola $\boxed{y^2 = x+1}$.

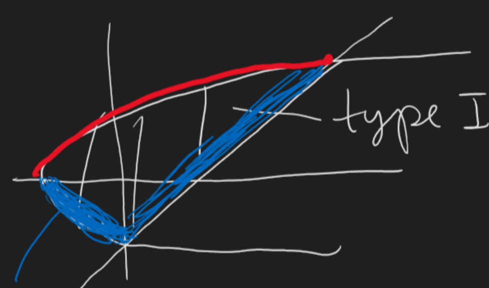



Solution

Poll: Is $D$ of type I or type II?

(a) I

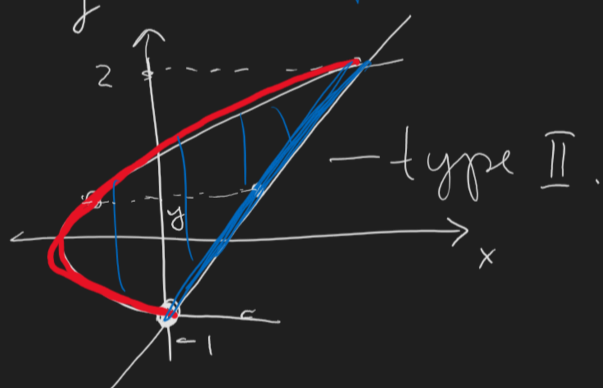
(b) II




As a type II region $D$ has easier description.

$g$

$g(x) = \begin{cases} -\sqrt{x+1}, & \ldots \\ \text{linear f-n}, & \ldots \end{cases}$




Need to find the pts of intersection:

$y^2 - 1 = y + 1$

$y^2 - y - 2 = 0 \Rightarrow y = -1, 2$

$y^2 - y - 2 = 0$

$$D = \{(x,y) \mid -1 \le y \le 2,\ y^2 - 1 \le x \le y+1\}.$$

$$\iint_D xy\, dA = \boxed{\int_{-1}^{2}\int_{y^2-1}^{y+1} xy\, dx\, dy} =$$

$$= \int_{-1}^{2} \left[ \frac{yx^2}{2} \right]_{x=y^2-1}^{x=y+1} dy = \int_{-1}^{2} \left( \frac{y(y+1)^2}{2} - \frac{y(y^2-1)^2}{2} \right) dy$$

$$= \ldots \qquad \square.$$

## Changing of order of integration

For a f-n on a rectangle

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

(Fubini).

For a f-n on a general, it is not the case!

Say you have a region of type I:

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx \neq \int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)\, dx\, dy$$

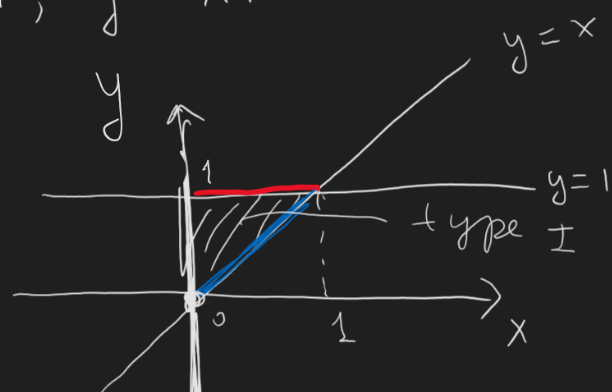
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \neq \int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)\,dx\,dy$$

(it is not even a number in the right-hand side!)

However if a region is of both types I and II, then you can change the order of integration, but you should change the limit of integration in correct way.

<u>Ex.</u> Find $\iint_D \sin(y^2)\,dA$, where $D$ is the region bounded by the lines $x=0$, $y=1$, $y=x$.

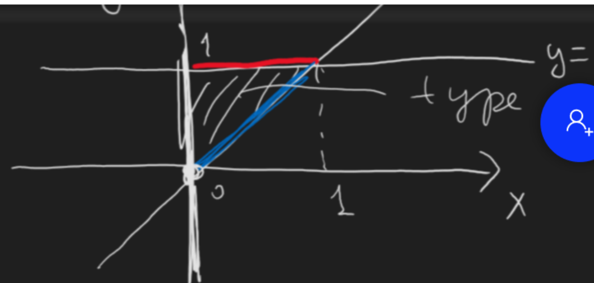
<u>Solution</u>



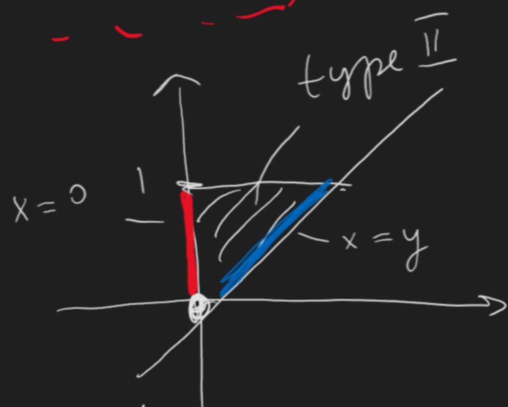

$$\iint_D \sin(y^2)\,dA = \int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$$

type II

Solution




$$\iint_D \sin(y^2)\,dA = \int_0^1 \boxed{\int_x^1 \sin(y^2)\,dy}\,dx$$




$$\iint_D \sin(y^2)\,dA = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy =$$

$$= \int_0^1 \Big[\sin(y^2)\cdot x\Big]_{x=0}^{x=y}\,dy =$$

$$= \int_0^1 \sin(y^2)\cdot y\,dy = \frac{1}{2}\int_0^1 \sin(y^2)\,d(\overset{t}{y^2}) =$$

$$= \frac{1}{2}\int_0^1 \sin t\,dt = \dots \quad \text{(easy)}. \quad \square$$

## Properties of double integrals.

1) $$\iint_D \big(f(x,y) + g(x,y)\big)\,dA = \iint_D f(x,y)\,dA + \iint g(x,y)\,dA$$

# Properties of double integrals

1) $$\iint_D \left(f(x,y) + g(x,y)\right) dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

(Indeed, $$\iint_D (f+g)\, dA = \iint_R (\tilde{f}+\tilde{g})\, dA = \lim \sum\sum \left(\tilde{f}(x_i^*, y_i^*) + \tilde{g}(x_i^*, y_i^*)\right) \Delta x \Delta y =$$

$$= \lim \sum\sum \tilde{f}(x_i^*, y_i^*) \Delta x \Delta y + \lim \sum\sum \tilde{g}(x_i^*, y_i^*) \Delta x \Delta y =$$

$$= \iint_D f\, dA + \iint_D g\, dA.$$

So this property of $\iint$ follows from the corresponding property of limits).

2) $$\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA.$$

3) If $f(x,y) \geq g(x,y)$ for all $(x,y)$ in $D$, then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$.

4) If $D = D_1 \cup D_2$, which don't overlap except possibly on their

$$+ \iint_D g(x,y)\, dA$$

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$= \lim \sum\sum \tilde{f}(x_i^*, y_i^*) \Delta x \Delta y + \lim \sum\sum \tilde{g}(x_i^*, y_i^*) \Delta x \Delta y =$

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3) If $f(x,y) \geq g(x,y)$ for all $(x,y)$ in $D$, then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$.

4) If $D = D_1 \cup D_2$, which don't overlap except possibly on their boundaries. Then


$$\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$$