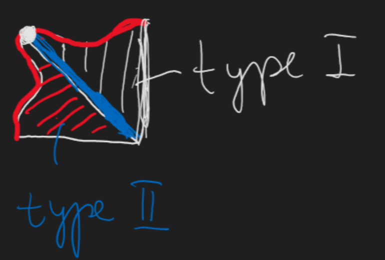
4) $D = D_1 \cup D_2$ which don't overlap except maybe on their boundaries

Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

If you have a region which is neither of type I nor of type II.

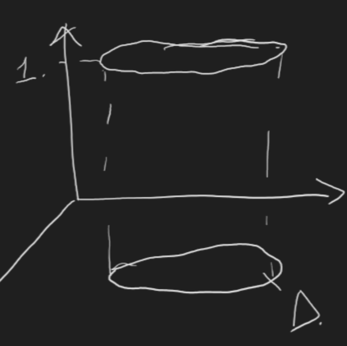


$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$

you know how to compute.

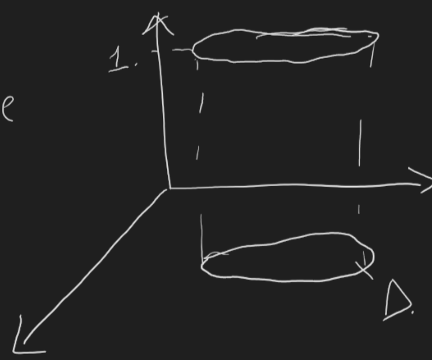
5) $\displaystyle\iint_D 1\,dA$ = the area of D

(Indeed $\displaystyle\iint_D 1\,dA$ = the volume of the cylinder of height 1 and with the base D =



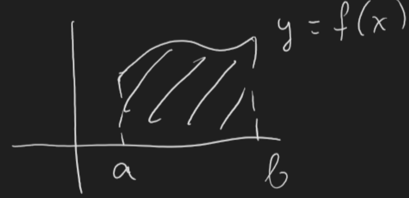
5) $\iint_D 1\, dA$ = the area of D

(Indeed $\iint_D 1\, dA$ = the volume of the cylinder of height 1 and with the base D = = the area of D).



We also have another formula for areas:

area below the graph = $= \int_a^b f(x)\, dx$.



But the formula $\iint_D 1\, dA$ = area of D is more general (works for any region).

Computing volumes.

Ex. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Computing volumes.

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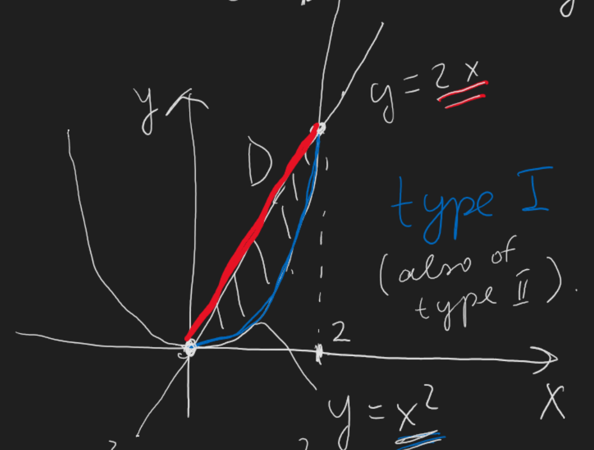
Solution



Points of intersection:

$2x = x^2$

$x(x-2) = 0$

$x = 0,\ x = 2.$

$D = \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$

$z = x^2 + y^2$ ~ the graph of $f(x,y) = x^2 + y^2$.

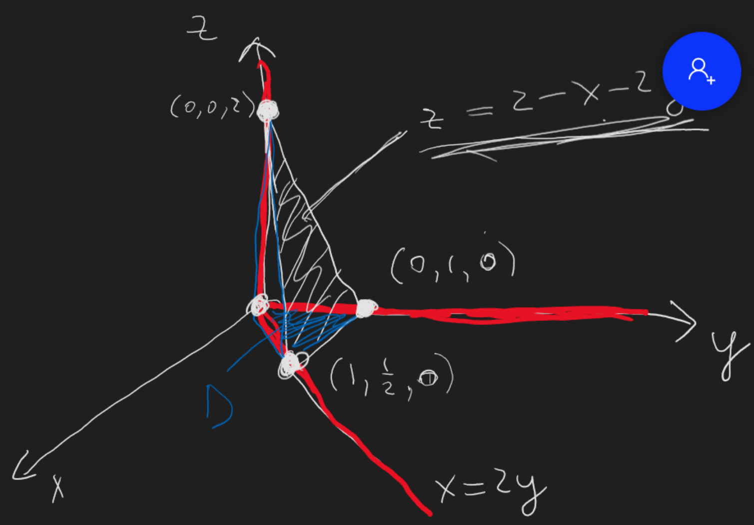
$$\text{the volume} = \iint_D (x^2+y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx =$$

$$= \dots \text{ (easy)}. \quad \square$$

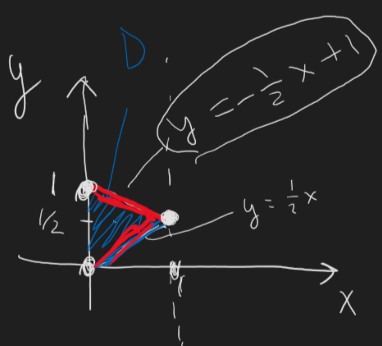
Ex. Find the volume of the tetrahedron bounded by the planes

$x + 2y + z = 2$, $x = 2y$, $x = 0$, $z = 0$.

Solution



volume of the tetrahedron = the volume under the graph of $f(x,y) = 2-x-2y$ above the plane region D. $= \iint_D (2-x-2y)\,dA.$



Will use description of D as of type I region

$$D = \{(x,y) \mid 0 \le x \le 1,\ \tfrac{1}{2}x \le y \le -\tfrac{1}{2}x + 1\}$$

Red line goes through pts $(0,1)$ and $(1,\tfrac{1}{2})$.

$y = ax + b$

$$\begin{aligned} 1 &= b \\ \tfrac{1}{2} &= a + b \end{aligned} \Rightarrow \begin{aligned} b &= 1 \\ a &= -\tfrac{1}{2} \end{aligned}$$

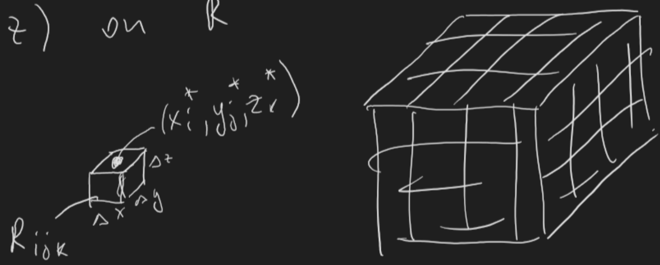
$$\iint_D (2-x-2y)\,dA = \int_0^1 \int_{\frac{1}{2}x}^{-\frac{1}{2}x+1} (2-x-2y)\,dy\,dx = \cdots \quad \square$$

Triple integrals over general regions

Triple integrals over rectangular boxes we already know:

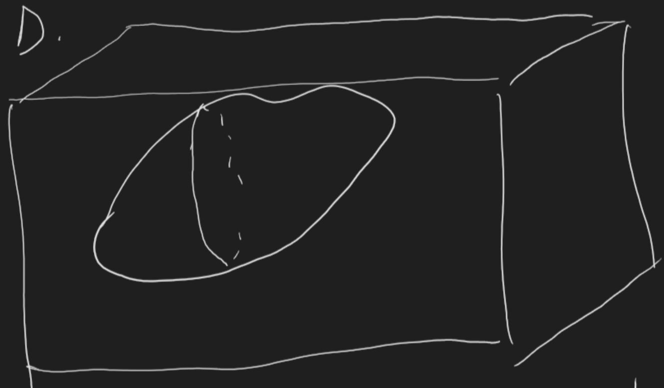
$f(x,y,z)$ on R

$\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$



$$\iiint_R f(x,y,z)\,dV \overset{\text{def}}{=} \lim_{n,m,l\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(x_i^*, y_j^*, z_k^*)\,\Delta V$$

Now we have $f(x,y,z)$ on a bounded region D.



Enclose D into a rectangular box R.

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \text{ is in } D \\ 0, & \text{if } (x,y,z) \text{ is in } R \text{ but not in } D. \end{cases}$$

Def. $\iiint_D f(x,y,z)\,dV = \iiint_R \tilde{f}(x,y,z)\,dV.$

(equivalent)
Alternative def.: $\iiint f(x,y,z)\,dV =$

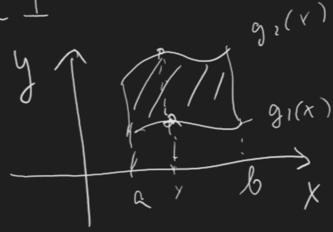
Def. $\iiint_D f(x,y,z)\,dV = \iiint_R \tilde{f}(x,y,z)\,dV$.

(Equivalent)
Alternative def.: $\iiint_D f(x,y,z)\,dV =$

$$= \lim_{n,m,l\to\infty} \sum_{\substack{i,j,k \\ \text{such that} \\ R_{ijk} \text{ lies in } D}} f(x_i^*, y_j^*, z_k^*)\,\Delta V.$$

How to compute $\iiint$?

type I

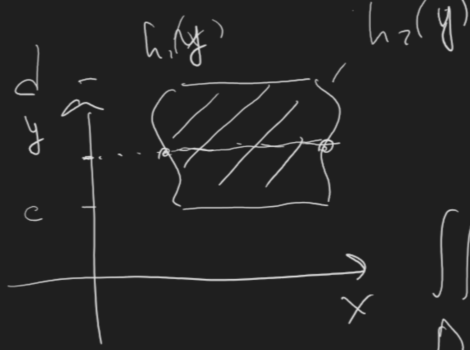


$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

outer

inner

type II



$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

outer

inner

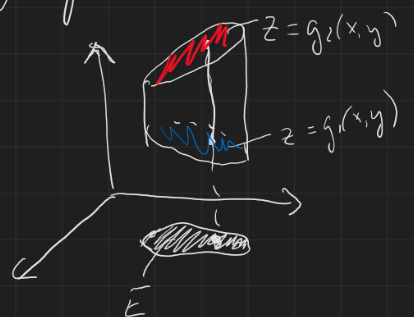
The same principle for $\iiint$:

choose 2 outer variables and determine in what range they can change. Then look in what limits the 3^{rd} (the inner one) variable can change.

The same principle for $\iiint$:

choose 2 outer variables and determine in what range they can change. Then look in what limits the 3^{rd} (the inner one) variable can change.

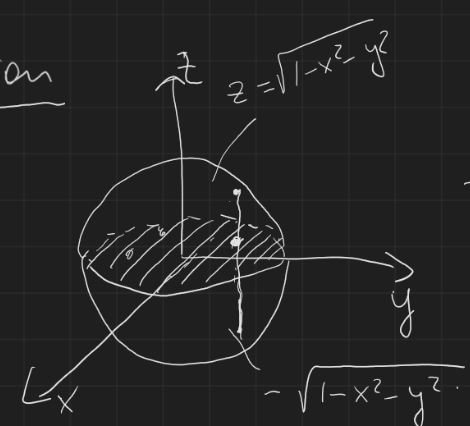
For example, if D is the solid between 2 graphs of f-s of x, y, then it makes sense to choose x, y as outer variables and z as inner variables.



$$\iiint_D f(x,y,z)\,dV = \iint_E \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) dA$$

<u>Ex.</u> Find $\iiint_D \underline{x}\, dV$, where D is the ball $x^2+y^2+z^2 \le 1$.

<u>Solution</u>



x, y – outer variables

z – inner variable.

sphere: $x^2+y^2+z^2=1$

upper hemisphere:

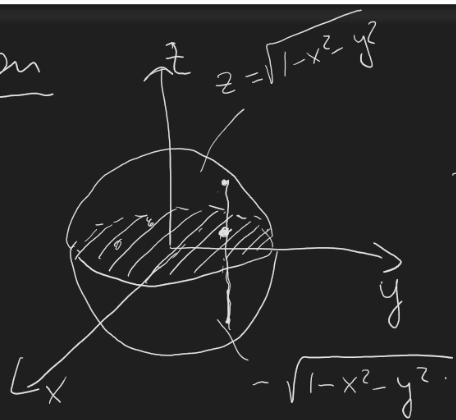
$$z = \sqrt{1-x^2-y^2}$$

Lower hemisphere

$$z = -\sqrt{1-x^2-y^2}$$

Microsoft Whiteboard

Solution



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upper hemisphere:

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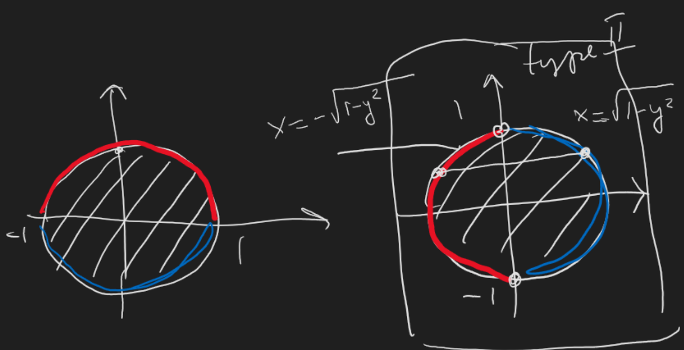
Lower hemisphere

$$z=-\sqrt{1-x^2-y^2}$$

$$\iiint_D x \, dV = \iint_{disc} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} x \, dz \, dA =$$

$$= \iint_{disc} \left[xz \right]_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} dA = \iint_{disc} 2x\sqrt{1-x^2-y^2} \, dA =$$



Poll: is it easier to integrate over x or over y?

(a) x

(b) y

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2x\sqrt{1-x^2-y^2} \, dx \, dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2} \, d(\overset{t}{x^2}) \, dy =$$

$$= \int_{-1}^{1} \int_{1-y^2}^{1-y^2} \sqrt{1-y^2-t} \, dt \, dy = 0. \qquad \square$$

0

$$\boxed{\iiint_D 1 \, dV = \text{the volume of } D.}$$

(Indeed, $\iiint_D 1 \, dV \approx \sum_{\substack{\text{all boxes} \\ \text{that lie} \\ \text{in } D.}} 1 \cdot \Delta V \approx$ the volume of D).

We also have the formula

$$\text{volume below the graph of } f(x,y) \text{ above region } D = \iint_D f(x,y) \, dA$$

The formula with $\iiint$ is more general – it works for any solid, while the formula with $\iint$ is only for solids that lie below the graphs of f-s.

it is not a solid lying below a graph of a f-n

So the 2nd formula doesn't apply.

here both formulas apply