How to change to cylindrical coordinates in triple integrals:

$$\iiint_D f(x,y,z)\,dV = \iiint_S f(r\cos\theta, r\sin\theta, z)\; r\,dr\,d\theta\,dz$$

the corresponding region in the $r\theta z$-space

Ex. Find $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the solid lying within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$.

Solution The solid is bounded by $r=1$, $z=4$, $z=1-r^2$.

$$E = \{(x,y,z) \mid x^2+y^2 \le 1,\ 1-x^2-y^2 \le z \le 4\} =$$

$$= \{(r,\theta,z) \mid r \le 1,\ 1-r^2 \le z \le 4,\ 0 \le \theta \le 2\pi\}.$$

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^1\int_0^{2\pi}\int_{1-r^2}^{4} r\cdot r\,dz\,d\theta\,dr = \dots\ (\text{easy}).$$

$\square$

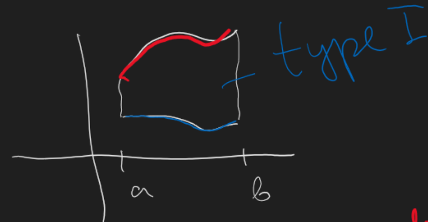
Section 15.2 : 12, 15, 31, 53, 58

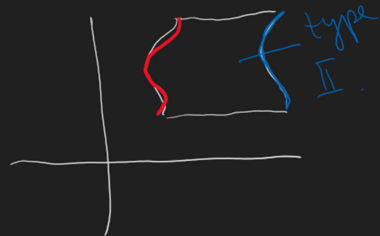
true - false test : 1 - 5.

12. Draw an example of a region that is (a) both type I and type II.

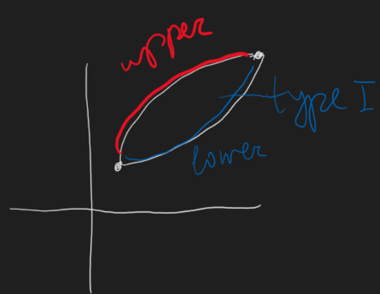
(b) neither type I nor type II.

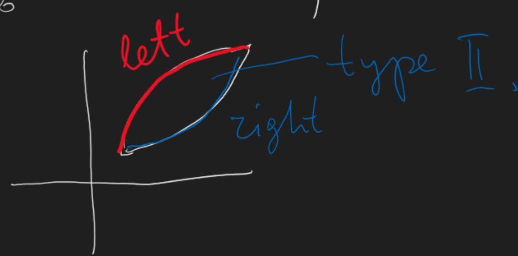
Solution



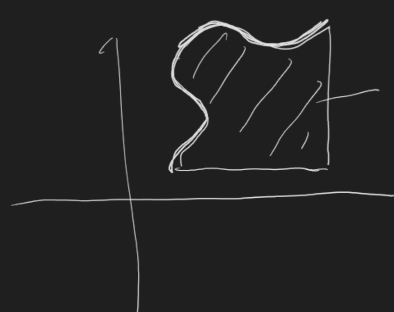


(a)





(b)



neither.

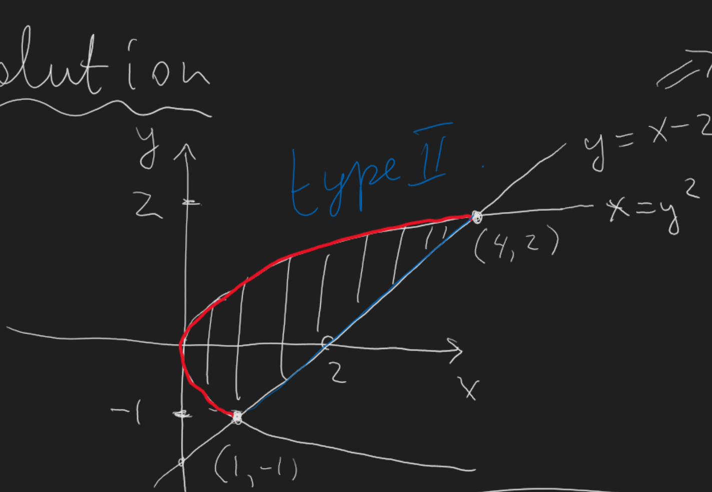
15. Set up iterated integral for both orders of integration. Then evaluate the double integral using the easier order and explain why it is easier.

$\iint y \, dA$, where D is bounded by

15 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it is easier.

$$\iint_D y\, dA, \quad \text{where } D \text{ is bounded by } y = x-2,\ x = y^2.$$

Solution



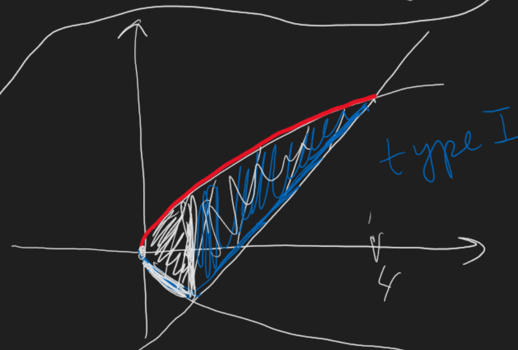
Points of intersection

$y = y^2 - 2$

$y^2 - y - 2 = 0$

$y = -1, 2.$

$$\iint_D y\, dA \underset{\text{using that } D \text{ is type II}}{=} \int_{-1}^{2}\int_{y^2}^{y+2} y\, dx\, dy \quad \text{— easy.}$$



$$D = \left\{(x,y) \,\middle|\, 0 \le x \le 4,\ \begin{cases} -\sqrt{x}, & x \le 1 \\ x-2, & x \ge 1 \end{cases} \le y \le \sqrt{x} \right\} =$$

$$= \{(x,y) \mid 0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x}\} \cup \{(x,y) \mid 1 \le x \le 4,\ x-2 \le y \le \sqrt{x}\}$$

$$\int_{\ }^{1}\int^{\sqrt{x}} \ldots \qquad \int^{4}\int^{\sqrt{x}} \ldots$$

$$D = \left\{ (x,y) \;\middle|\; 0 \le x \le 4,\ \begin{cases} -\sqrt{x}, & x \le 1 \\ x-2, & x \ge 1 \end{cases} \le y \le \sqrt{x} \right\} =$$

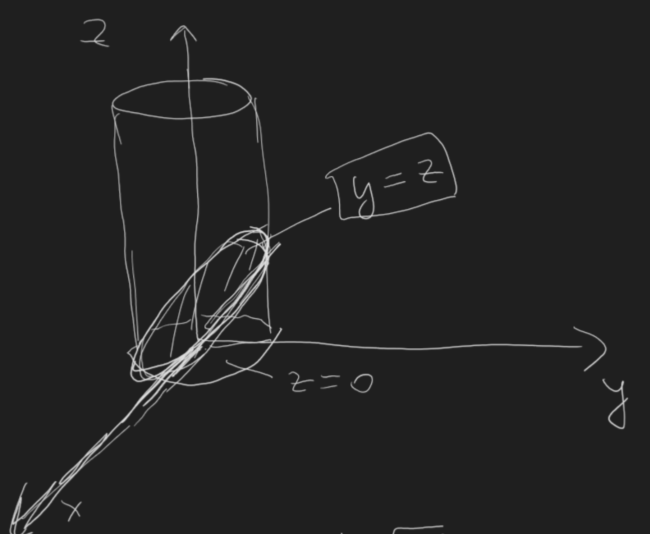
$$= \underbrace{\left\{ (x,y) \;\middle|\; 0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x} \right\}} \cup \left\{ (x,y) \;\middle|\; 1 \le x \le 4,\ x-2 \le y \le \right.$$

$$\boxed{\iint_D y\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\, dy\, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\, dy\, dx}$$

more complicated.

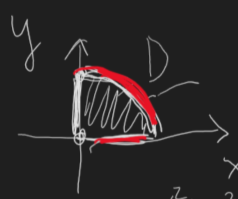
31. Find the volume of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $y=z$, $\boxed{x=0}$, $z=0$ in the first octant.

Solution



$$\text{the volume} = \iint_D y\, dA = \int_0^1 \int_0^{\sqrt{1-y^2}} y\, dx\, dy =$$

$$= \int_0^1 \left[yx \right]_{x=0}^{x=\sqrt{1-y^2}} dy =$$



$x^2+y^2=1,\ x=\sqrt{1-y^2}$

$$= \int_0^1 \left[y x\right]_{x=0}^{x=\sqrt{1-y^2}} dy =$$

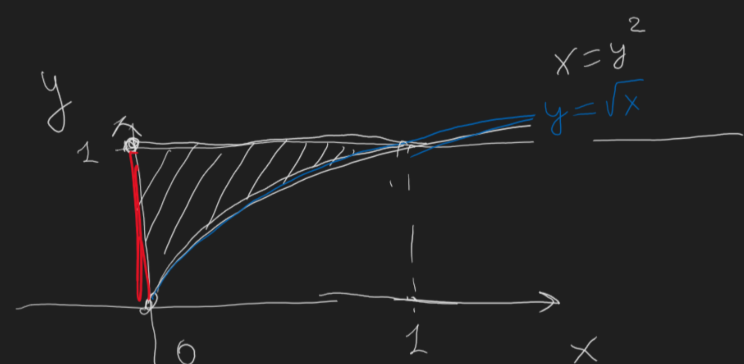
$$= \int_0^1 y\sqrt{1-y^2}\,dy = \frac{1}{2}\int_0^1 \sqrt{1-t}\,dt = -\frac{1}{2}\int_1^0 \sqrt{u}\,du =$$

$$= \left[-\frac{1}{2}\cdot u^{\frac{3}{2}}\cdot\frac{2}{3}\right]_{u=1}^{u=0} = \frac{1}{3}. \qquad \square$$

53. Evaluate the integral by reversing the order of integration.

$$\int_0^1\int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx.$$

Solution



$$\int_0^1\int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \left[\sqrt{y^3+1}\cdot x\right]_{x=0}^{y^2} dy =$$

$$= \int_0^1 y^2\sqrt{y^3+1}\,dy = \frac{1}{3}\int_1^2 \sqrt{t}\,dt = \left[\frac{1}{3}\cdot\frac{2}{3}\,t^{3/2}\right]_{t=1}^{t=2} =$$

$$= \frac{2}{9}\left(2\sqrt{2}-1\right). \qquad \square$$

58. Express D as a union of regions

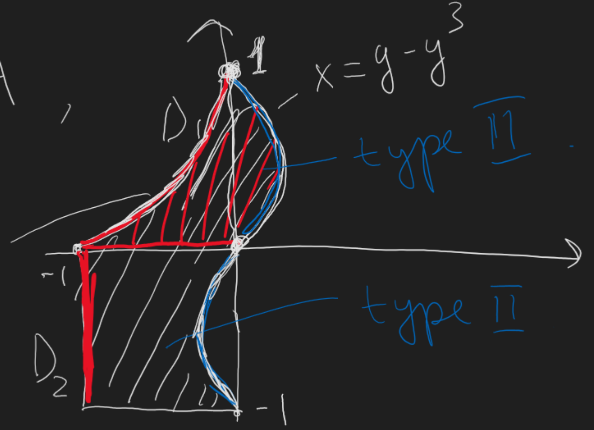
58. Express D as a union of regions of type I or type II and evaluate the integral.

$$\iint_D y \, dA,$$

$y=(x+1)^2$

$x+1=\sqrt{y}$

$x=\sqrt{y}-1$



Solution: D is neither type I nor type II.

$D = D_1 \cup D_2$

$D_1 = \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y}-1 \le x \le y-y^3\},$

$D_2 = \{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y-y^3\}$

$$dA = \iint_{D_1} y\,dA + \iint_{D_2} y\,dA =$$

$$= \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy + \int_{-1}^{0} \int_{-1}^{y-y^3} y\,dx\,dy =$$

$$= \int_0^1 [yx]_{x=\sqrt{y}-1}^{x=y-y^3} dy + \int_{-1}^{0} [yx]_{x=-1}^{x=y-y^3} dy =$$

$$\ldots \int_{-1}^{0} \left(y(y-y^3)+y\right)dy$$

$0 \quad \sqrt{y}-1$

$$= \int_0^1 [y\,x]_{x=\sqrt{y}-1}^{x=y-y^3} \, dy + \int_{-1}^0 [y\,x]_{x=-1}^{x=y-y^3} \, dy$$

$$= \int_0^1 \left(y(y-y^3) - y(\sqrt{y}-1)\right) dy + \int_{-1}^0 \left(y(y-y^3) + y\right) dy$$

$$= \int_{-1}^1 y(y-y^3)\, dy - \int_0^1 y(\sqrt{y}-1)\, dy + \int_{-1}^0 y \, dy =$$

$= \dots$ easy.

True-false test: 1 – 5.

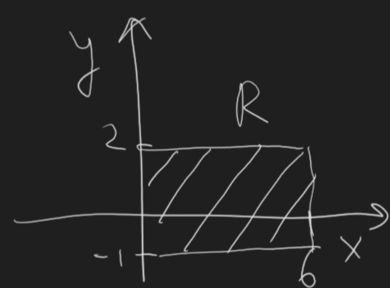
Poll:

(a) True

(b) False.

1. $$\int_{-1}^{2}\int_{0}^{6} x^2 \sin(x-y)\, dx\, dy = \int_{0}^{6}\int_{-1}^{2} x^2 \sin(x-y)\, dy\, dx$$

True: those iterated integrals $= \iint_R x^2 \sin(x-y)\, dA$



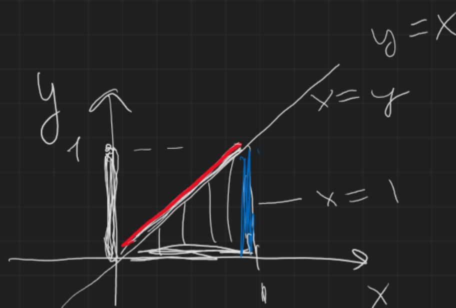
When integrating over rectangle, both orders are fine, by Fubini's Theorem.

2. $$\int_0^1 \int_0^x \sqrt{x+y^2}\, dy\, dx = \int_0^x \int_0^1 \sqrt{x+y^2}\, dx\, dy$$

2. $\displaystyle\int_0^1\int_0^x \sqrt{x+y^2}\,dy\,dx = \int_0^1\int_0^x \sqrt{x+y^2}\,dx\,dy$

False: The right-hand side doesn't make sense, it is not even a number.

Should be:



$$\int_0^1\int_0^x \sqrt{x+y^2}\,dy\,dx =$$

$$= \int_0^1\int_y^1 \sqrt{x+y^2}\,dx\,dy.$$

3. $\displaystyle\int_1^2\int_3^4 x^2 e^y\,dy\,dx = \int_1^2 x^2\,dx \cdot \int_3^4 e^y\,dy$

True: $\displaystyle\int_1^2\int_3^4 x^2 e^y\,dy\,dx = \int_1^2 x^2 \int_3^4 e^y\,dy\,dx =$

const, can put it outside the integral

just a const

$$= \int_3^4 e^y\,dy \cdot \int_1^2 x^2\,dx.$$

4. $\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2}\sin y\,dx\,dy = 0$

True

~~False~~: $\int_{-1}^{1}\int_{0}^{1} e^{x^2}\cdot e^{y^2}\sin y\,dx\,dy =$

$= \int_{-1}^{1} e^{y^2}\sin y\left(\int_{0}^{1} e^{x^2}dx\right)dy =$

$= \boxed{\int_{0}^{1} e^{x^2}dx}\cdot\int_{-1}^{1} e^{y^2}\sin y\,dy$

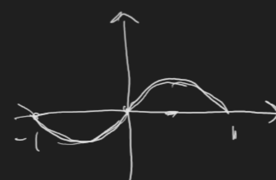
even f-n

odd f-n

odd f-n

$0 <$

$= 0.$



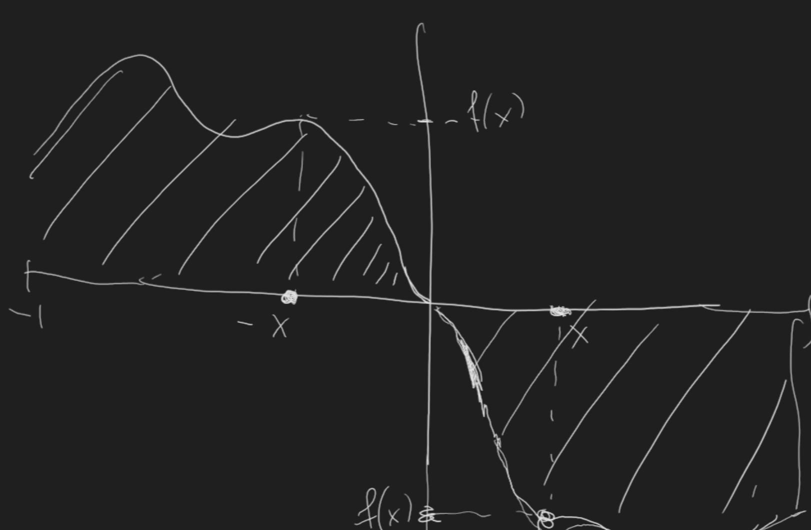
$y > 0$: $e^{y^2}\sin y$

$-y$: $e^{y^2}\sin(-y) = -e^{y^2}\sin y$

odd.

even

Average value of such f-n at [-1, 1] is 0.



False: $\int_{-1}^{1}\int_{0}^{1} e^{x^2} \cdot e^{y^2} \sin y \, dx \, dy =$

$$= \int_{-1}^{1} e^{y^2} \sin y \left(\int_{0}^{1} e^{x^2} \, dx \right) dy =$$

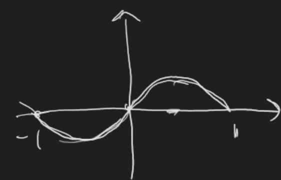
$$= \boxed{\int_{0}^{1} e^{x^2} dx} \cdot \int_{-1}^{1} \underbrace{e^{y^2}}_{\text{even f-n}} \underbrace{\sin y}_{\text{odd f-n}} \, dy$$

$0 <$

odd f-n

$\| $

$0.$



even

$y > 0$: $e^{y^2} \sin y$

$-y$: $e^{y^2} \sin(-y) = -e^{y^2} \sin y$

odd.

Average value of such f-n at $[-1, 1]$ is 0.