Change of variables.

For f-s of 1 variable:

$$\int_a^b f(\underbrace{g(x)}_{t})\,\underbrace{g'(x)\,dx}_{dt} = \int_{g(a)}^{g(b)} f(t)\,dt$$

$$\iint_D \underbrace{(x^2+y^2)}_{u}^{7/2}\,\underbrace{(x^2-y^2)}_{v}^{19}\,\overbrace{dx\,dy}^{dA}$$

How to express in terms of u, v?

How to express in terms of u, v?

$\boxed{x = g(u,v)}$, $\boxed{y = h(u,v)}$.

One says that we have a transformation T from the uv-plane to the xy-plane.

Our assumptions about T:

1) T is C^1-transformation, meaning that g and h have continuous partial derivatives.

2) T is 1-to-1, meaning that it send different points to

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1) T is C^1-transformation, meaning that g and h have continuous partial derivatives.

2) T is 1-to-1, meaning that it send different points to differents points.

Simply speaking, it means we can express u, v in terms of x, y.

Def. The Jacobian of the tranformation T given by $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} =$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Remark In the book det is denoted by $|\ |$ — can be confused with the abs. value.

So we use det.

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Remark In the book det is denoted by $|\ |$ — can be confu[...] the abs. value.

So we use det.

Change of variables in double integrals:

Suppose that T maps a region S in the uv-plane onto a region D in the xy-plane. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Ex. Find $\iint_D e^{\frac{x+y}{x-y}}\,dA$, where D is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

$u = x+y$, $v = x-y$

Solution We make change of variables: $u = x+y$, $v = x-y$. To find the Jacobian, we at first express x, y in terms of u, v.

$2x = u+v$, $2y = u-v$

$x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

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Solution: We make change of variables:

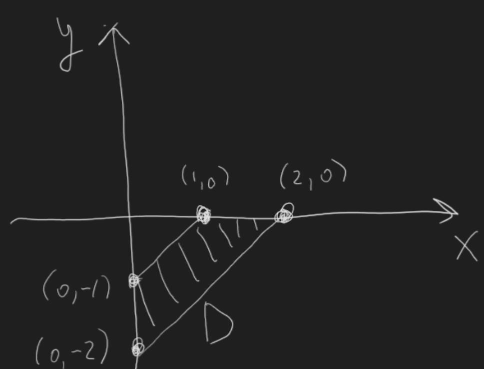
$\underline{u = x+y}$, $\underline{v = x-y}$. To find [the] Jacobian, we at first express x, y in terms of u, v.

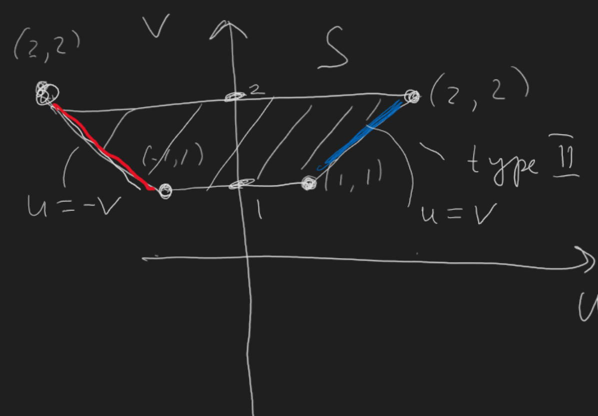
$2x = u+v$, $2y = u-v$

$x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} =$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$





$$\iint_D e^{\frac{x+y}{x-y}}\, dA = \iint_S e^{\frac{u}{v}} \left|-\tfrac{1}{2}\right| du\, dv =$$

$$= \frac{1}{2} \iint_S e^{\frac{u}{v}}\, du\, dv = \frac{1}{2} \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv =$$

$$= \frac{1}{2} \int_1^2 \left[v e^{\frac{u}{v}} \right]_{u=-v}^{u=v} dv = \frac{1}{2} \int_1^2 \left(ve - ve^{-1} \right) dv =$$

$$= \frac{1}{2} \int_1^2 v\left(e - e^{-1}\right) dv = \frac{1}{2} \left[\frac{v^2}{2} \left(e - e^{-1}\right) \right]_{v=1}^{v=2} =$$

$$= \frac{1}{2}\int_1^2 v(e-e^{-1})\,dv = \frac{1}{2}\left[\frac{v^2}{2}(e-e^{-1})\right] =$$

Now about triple integrals:

$x = g(u,v,w)$, $y = h(u,v,w)$, $z = k(u,v,w)$

– transformation T from the uvw-space to the xyz-space.

Def. The **Jacobian** of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Formula for changing variables in triple integrals:

$$\boxed{\iiint_D f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

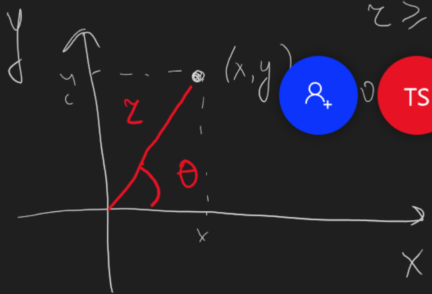
Will apply to:

- Polar coordinates
- Cylindrical coordinates
- Spherical coordinates.

Polar coordinates y (x,y) $r \geq 0$ $0 \leq \theta \leq 2\pi$

Polar coordinates

r, θ – polar coordinates.



$r \geq 0$

$$\boxed{x = r\cos\theta}, \quad y = r\sin\theta$$

$$\Rightarrow x^2 + y^2 = r^2$$

The Jacobian:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} =$$

$$= \cos\theta \cdot r\cos\theta + r\sin\theta \cdot \sin\theta = r(\cos^2\theta + \sin^2\theta) =$$
$$= r.$$

How to change to polar coordinates in double integrals:

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

S – the corresponding region in the $r\theta$-plane

When it makes sense to change to polar coordinates:

when f looks nice in polar coordinates

and

when the region looks nice in polar coordinates.

For example:

if $D = \{(x,y) \mid x^2+y^2 \leq 1\}$, then

For example:

- if $D=\{(x,y)\mid x^2+y^2\le 1\}$, then

$$S=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$$

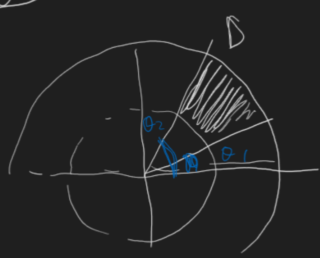
— a rectangle

- if D, then

$$S=\{(r,\theta)\mid R_1\le r\le R_2,\ 0\le\theta\le 2\pi\}$$

— a rectangle.

- if

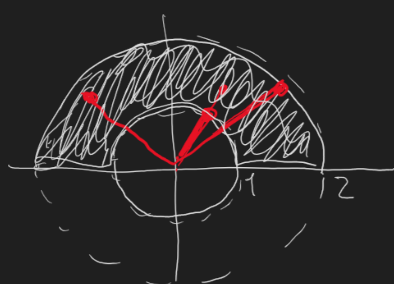


$$S=\{(r,\theta)\mid R_1\le r\le R_2,\ \theta_1\le\theta\le\theta_2\}$$

— a rectangle.

Ex. Find $\iint_D x^2(x^2+y^2)^7\,dA$, where D is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$

Solution



$$S=\{(r,\theta)\mid 1\le r\le 2,\ 0\le\theta\le\pi\}$$

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Solution

$$S = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

$$\iint_D x^2 (x^2+y^2)^7 \, dA = \iint_S r^2 \cos^2\theta \cdot r^{14} \, r \, dr \, d\theta =$$

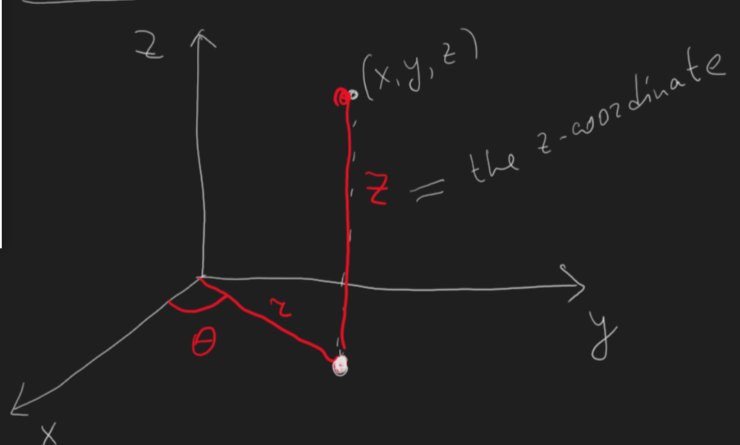
$$= \int_0^{\pi}\int_1^{2} r^{17} \cos^2\theta \, dr \, d\theta = \int_0^{\pi} \left[\frac{r^{18}}{18} \cdot \cos^2\theta \right]_{r=1}^{r=2} d\theta =$$

$$= \int_0^{\pi} \cos^2\theta \left(\frac{2^{18}-1}{18} \right) d\theta = \frac{2^{18}-1}{18} \int_0^{\pi} \cos^2\theta \, d\theta =$$

$$= \frac{2^{18}-1}{18} \int_0^{\pi} \frac{1+\cos 2\theta}{2} \, d\theta = \frac{2^{18}-1}{18} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_{\theta=0}^{\theta=\pi} =$$

$$= \frac{2^{18}-1}{18} \cdot \frac{\pi}{2}. \qquad \square$$

Cylindrical coordinates: r, θ, z



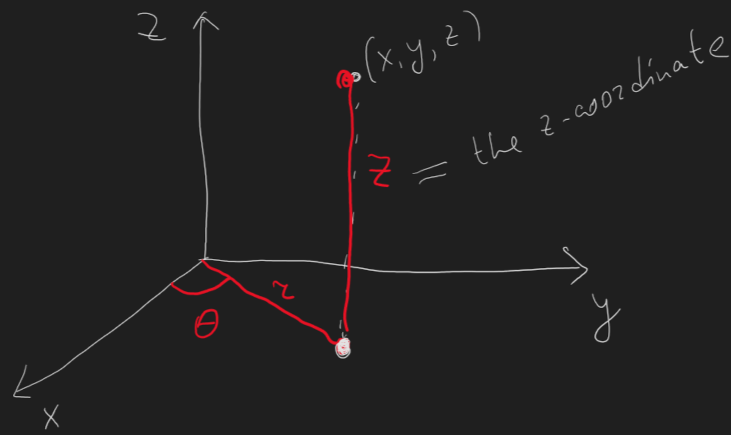
$r \ge 0$

$0 \le \theta \le 2\pi$

z can be any.

Why such name?

Cylindrical coordinates: r, θ, z



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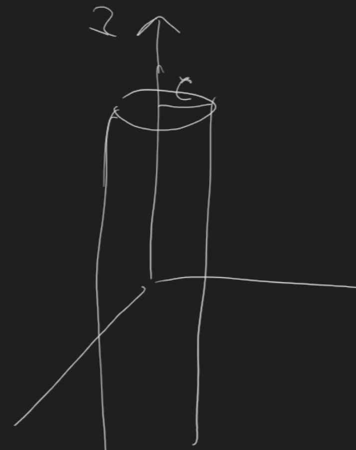

Equation of cylinder:

$$x^2 + y^2 = c^2$$

$$r^2 = c^2$$

$\boxed{r = c}$ — very easy form.

It might make sense to change to cylindrical coordinates when there is symmetry around the z-axis.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

The Jacobian:

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r.$$