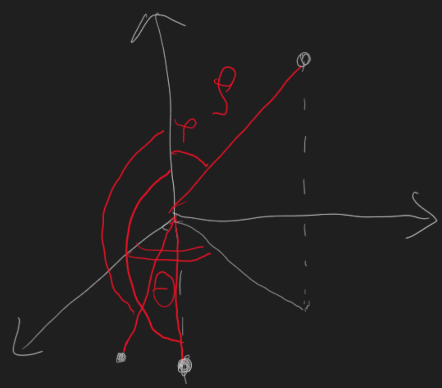
Previous lecture

spherical coordinates



φ – angle between the red line and the positive part of axis z.

$$\iiint_D f(x,y,z)\,dV = \iiint_S f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\cdot \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

Mass of a solid of density $\rho(x,y,z)$ =

$$= \boxed{\iiint_D \rho(x,y,z)\,dV.}$$

Center of mass (x_0, y_0, z_0) of a solid of density $\rho(x,y,z)$:

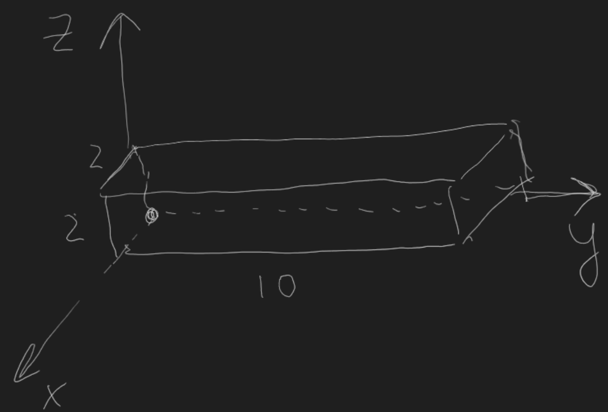
$$x_0 = \frac{\iiint_D x\,\rho(x,y,z)\,dV}{m}$$

$$y_0 = \frac{\iiint_D y\,\rho(x,y,z)\,dV}{m}$$

$$z_0 = \frac{\iiint_D z\,\rho(x,y,z)\,dV}{}$$

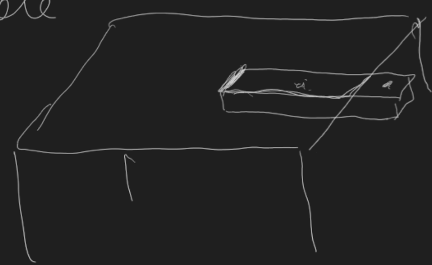
$$z_0 = \frac{\iiint_D z\, \rho(x,y,z)\, dV}{m}$$

Ex.



$\rho(x,y,z) = y^3$

The solid is put on a table such that $\frac{2}{3}$ of its length (counting from the left) lie on the table.



Determine whether the solid will fall down or not.

Solution (x_0, y_0, z_0) – center of mass.

(x_0, y_0, z_0) is above the table $\Longrightarrow$ doesn't fall.

is outside the table $\Longrightarrow$ falls down.

$y_0 \leq \frac{2}{3}$ of length $= \frac{2}{3} \cdot 10 = \frac{20}{3} \Longrightarrow$ doesn't fall.

$y_0 > \frac{20}{3} \Longrightarrow$ falls.

$$y_0 = \frac{\iiint_D y \cdot \rho(x,y,z)\, dV}{\iiint_D \rho(x,y,z)\, dV}$$

$$\iiint_D \rho(x,y,z)\,dV = \int_0^2\int_0^2\int_0^{10} y\cdot y^3\,dy\,dx\,dz =$$

$$= \int_0^2\int_0^2 \left[\frac{y^5}{5}\right]_{y=0}^{y=10} dx\,dz = \int_0^2\int_0^2 \frac{10^5}{5}\,dx\,dz = \frac{10^5}{5}\cdot 4 = \frac{4\cdot 10^5}{5}$$

$$\iiint_D \rho(x,y,z)\,dV = \int_0^2\int_0^2\int_0^{10} y^3\,dy\,dx\,dz = \int_0^2\int_0^2 \left[\frac{y^4}{4}\right]_{y=0}^{y=10} dx\,dz =$$

$$= \int_0^2\int_0^2 \frac{10^4}{4}\,dx\,dz = \frac{10^4}{4}\cdot 4 = 10^4.$$

$$y_0 = \frac{4\cdot 10^5}{5\cdot 10^4} = 8 > \frac{20}{3} \Longrightarrow$$

falls down. □

Remark. If you have a flat region of density $\rho(x,y)$, then

$$m = \iint_D \rho(x,y)\,dA, \ \ldots$$

Curves.

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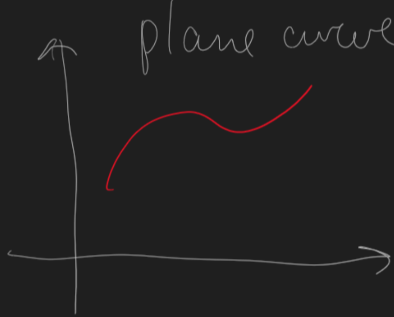
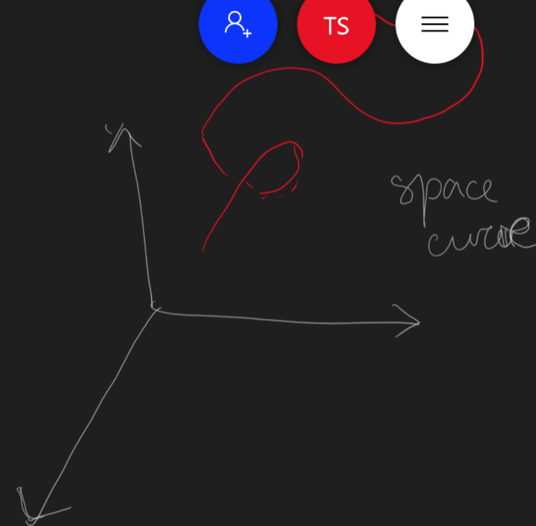
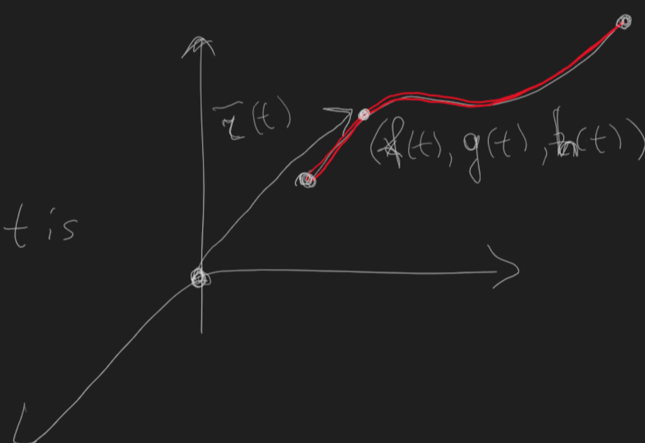




<u>Def</u>. Suppose f, g, h are continuous f-s. The set of all points $(f(t), g(t), h(t))$, where t is from an interval I, is <u>curve</u>.



Equivalently, a curve is represented by the position vector $\bar{r}(t) = \langle f(t), g(t), h(t) \rangle$
It a vector function.

$\bar{r}(t)$ is called a <u>parametrization</u> of a curve. t is called <u>parameter</u>.

A curve might have many different parametrizations.

<u>Examples</u> 1)

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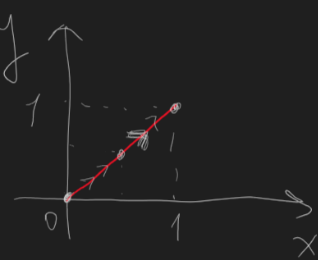


t = 0.5
(0.5, 0.5)

$\bar{r}(t) = \langle t, t \rangle, \quad 0 \le t \le 1.$

Another parametrization: $\bar{r}(t) = \langle 1-t, 1-t \rangle, \quad 0 \le t \le 1.$

Another parametrization:

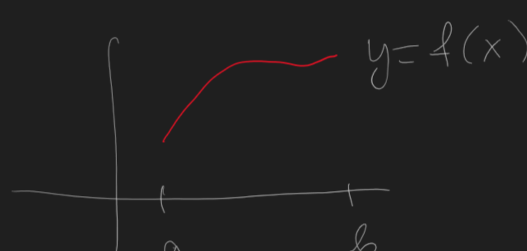
$\bar{r}(t) = \langle t^2, t^2 \rangle, \quad 0 \le t \le 1.$

t = 0.5
(0.25, 0.25)

2) circle

$\bar{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi.$

3) a graph of a f-n of 1 variable.



$\bar{r}(t) = \langle t, f(t) \rangle.$

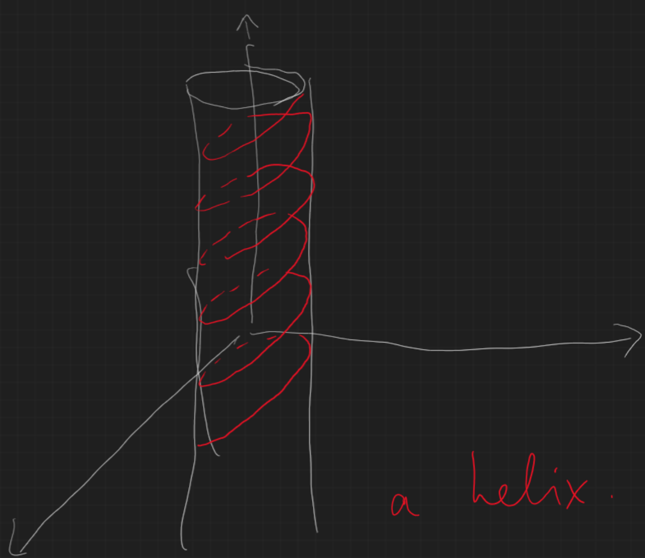
a b

$\bar{r}(t) = \langle t, f(t) \rangle$.

Ex. Sketch the curve whose $\overbrace{\text{vector equation}}^{\text{parametrization}}$ is $\bar{r}(t) = \langle \cos t, \sin t, t \rangle =$

$= \cos t\, \bar{i} + \sin t\, \bar{j} + t\, \bar{k}$.

Solution: $\cos^2 t + \sin^2 t = 1 \Rightarrow$ each point on our curve satisfies the eq-n of cylinder $x^2 + y^2 = 1$, so our curve lies on the cylinder.



a helix.

Ex: Find a parametrization of the curve of intersection of the cylinder $\boxed{x^2 + y^2 = 1}$ and the plane $\boxed{y + z = 2} \Rightarrow z = 2 - y$

z

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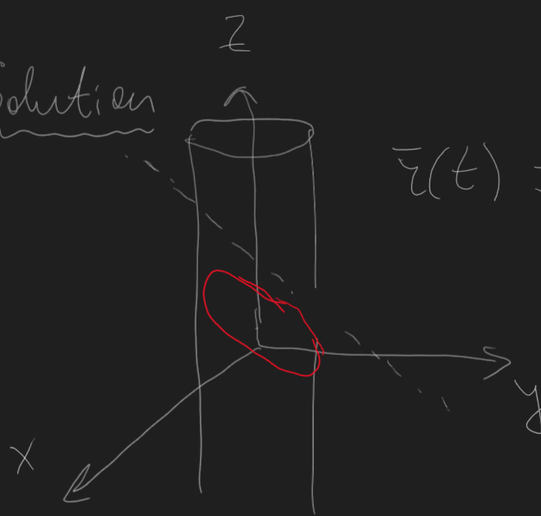
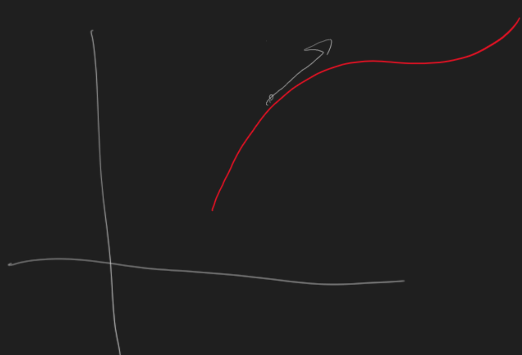
Solution



$$\bar{r}(t) = \langle \cos t, \sin t, 2-\sin t \rangle$$

$$0 \le t \le 2\pi.$$

□



How to differentiate vector f-s.

Def. $$\bar{r}'(t) = \lim_{h\to 0} \frac{\bar{r}(t+h)-\bar{r}(t)}{h}$$

It is a vector f-n.

How to compute?

Proposition If $\bar{r}(t) = \langle f(t), g(t), k(t) \rangle$, How to compute?

Proposition If $\bar{r}(t) = \langle f(t), g(t), k(t) \rangle$,

then $\bar{r}'(t) = \langle f'(t), g'(t), k'(t) \rangle$.

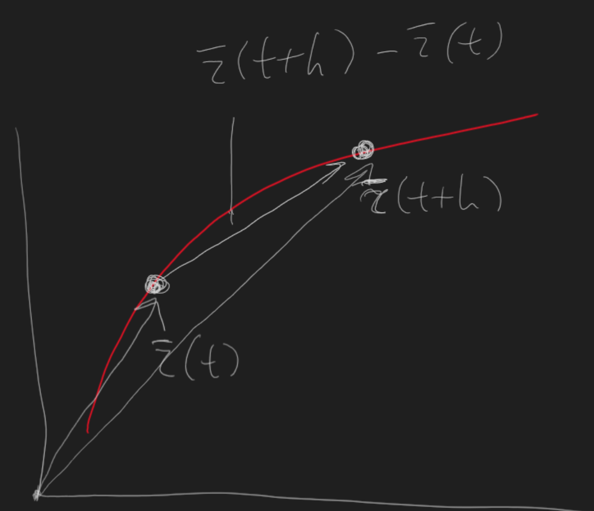
Proof $\bar{r}'(t) = \lim\limits_{h \to 0} \dfrac{\bar{r}(t+h) - \bar{r}(t)}{h} =$

$$= \lim_{h \to 0} \frac{\langle f(t+h) - f(t), g(t+h) - g(t), k(t+h) - k(t) \rangle}{h} =$$

$$= \lim_{h \to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{k(t+h) - k(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \to 0} \frac{k(t+h) - k(t)}{h} \right\rangle$$

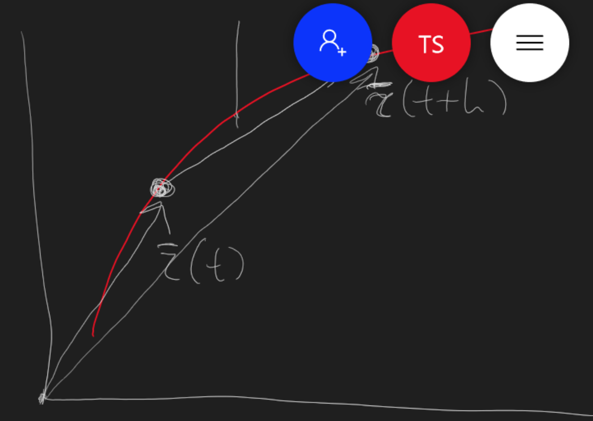
$$= \langle f'(t), g'(t), k'(t) \rangle. \quad \square$$



Geometrical sense.

$\bar{r}(t+h) - \bar{r}(t)$ is a secant vector

$\dfrac{\bar{r}(t+h) - \bar{r}(t)}{h}$ — a v-r going in the same direction

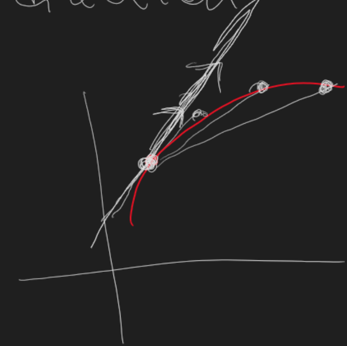


Geometrical sense.

$\bar r(t+h)-\bar r(t)$ is a secant vector

$\frac{\bar r(t+h)-\bar r(t)}{h}$ – a v-r going in the same direction

$\boxed{\lim_{h\to 0}\frac{\bar r(t+h)-\bar r(t)}{h}} = \bar r'(t)$ is a tangent v-r at pt corresponding to parameter value t.



<u>Def.</u> $\bar r'(t)$ is called <u>the tangent vector.</u>

(It depends on parametrization).

<u>Def.</u> <u>The unit tangent vector</u> is $\bar T(t) = \frac{\bar r'(t)}{|\bar r'(t)|}$.

$|\vec{r}\,'(t)|$

Ex. For the curve $\vec{r}(t) = \sqrt{t}\,\vec{i} + (2-t)\,\vec{j}$, find the unit tangent vector at the point where $t=1$.

Solution $\quad \vec{T}(1) = \dfrac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|}$.

$$\vec{r}(t) = \langle \sqrt{t},\ 2-t \rangle$$

$$\vec{r}\,'(t) = \left\langle \frac{1}{2\sqrt{t}},\ -1 \right\rangle$$

$$\vec{r}\,'(1) = \left\langle \frac{1}{2},\ -1 \right\rangle$$

$$|\vec{r}\,'(1)| = \sqrt{\frac{1}{4}+1} = \frac{\sqrt{5}}{2}.$$

$$\vec{T}(1) = \frac{\left\langle \frac{1}{2}, -1 \right\rangle}{\frac{\sqrt{5}}{2}} = \left\langle \frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}} \right\rangle. \qquad \square$$

Ex. Find parametric equation for the tangent line to the helix given by parametrization $x = 2\cos t$, $y = \sin t$, $z = t$ at the point $(0, 1, \frac{\pi}{2})$.

Equation of a line: $\cancel{\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b}} = \dfrac{z-z_0}{c}$.

$$\boxed{\vec{r} = \vec{r}_0 + t\,\vec{v}}$$

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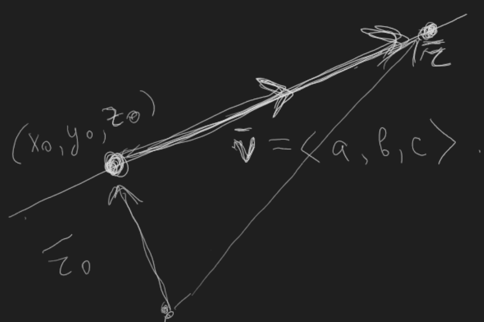

$$\boxed{\bar{r} = \bar{r}_0 + t\bar{v}}$$

– vector equation of a line.

parametric equations of a line:

$$\boxed{\begin{aligned} x(t) &= x_0 + ta \\ y(t) &= y_0 + tb \\ z(t) &= z_0 + tc \end{aligned}}$$

Solution: At first we find the parameter value corresponding to $(0, 1, \frac{\pi}{2})$.

Since $z = t \implies \boxed{t = \frac{\pi}{2}}$

Tangent line: goes through $(0, 1, \frac{\pi}{2})$ in the direction of the tangent vector $\bar{r}'(\frac{\pi}{2})$.

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Tangent line: goes through $(0, 1, \frac{\pi}{2})$ in the direction of the tangent vector $\bar{r}'(\frac{\pi}{2})$.

$$\bar{r}(t) = \langle 2\cos t, \sin t, t \rangle$$

$$\bar{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle.$$

$$\bar{r}'(\tfrac{\pi}{2}) = \boxed{\langle -2, 0, 1 \rangle}$$

The parametric equations of the tangent line:

$$x(t) = -2t$$

$$y(t) = 1$$

$$z(t) = \frac{\pi}{2} + t. \qquad \square$$