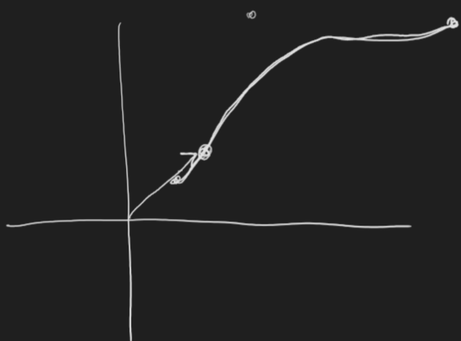
Previous lecture

Curves

$$\bar{r}(t) = \langle x(t), y(t) \rangle$$



$$\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$$

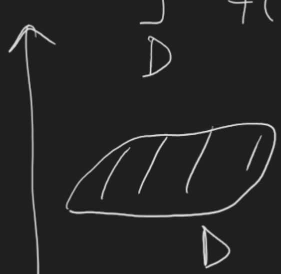
— parametrization.

$\bar{r}'(t) = \langle x'(t), y'(t) \rangle$ — the tangent v-r at the pt corresponding to the parameter value t.

$$\bar{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}$$ — the unit tangent vector.

Line integrals (integrals over curves)

$$\int_D f(x,y)\, dA$$



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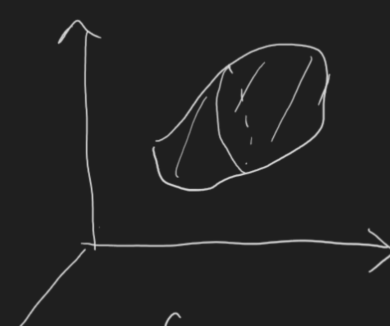
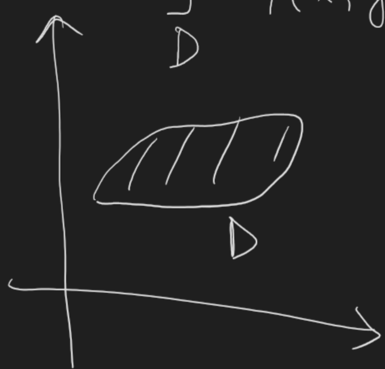
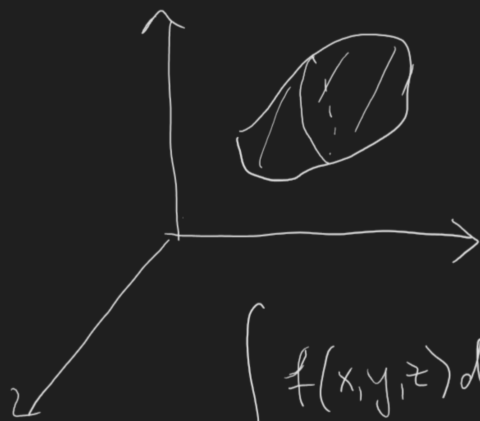


Line integrals (integrals over curves)

$$\int_D f(x,y)\, dA$$

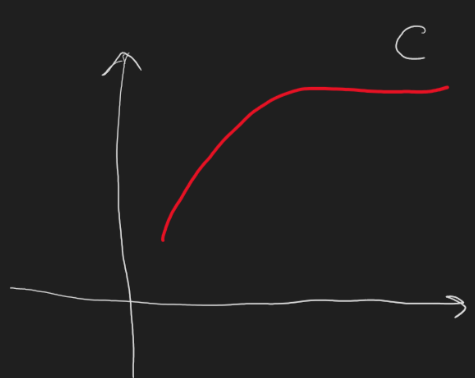




$$\int_D f(x,y,z)\, dV$$

Today will learn how to integrate

- f-s of 2 variables over plane curves
- f-s of 3 variables over space curves.



Suppose C is given by

$\vec{r}(t) = \langle x(t), y(t) \rangle$)

$a \le t \le b$

Assumption about C: C is smooth meaning that $\vec{r}\,'(t)$ $\exists$ and $\neq 0$.

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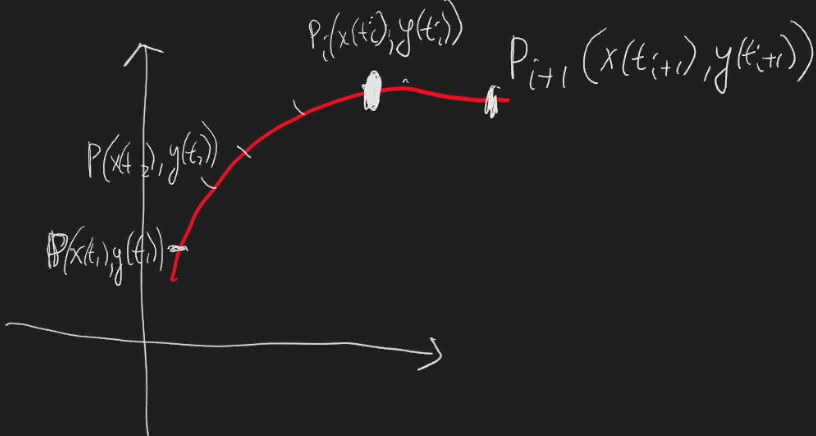
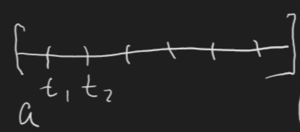
meaning that $\vec{r}\,'(t)$ $\exists$ and $\neq 0$

Suppose $f(x,y)$ is defined on C.

(Can be defined also on $\mathbb{R}^2$, but only its values on C matter for us).



Divide $[a,b]$ into n subintervals of length Δt.



$$\Delta t = \frac{b-a}{n}.$$

C is divided into n subarcs of length $\boxed{\Delta s_i}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta s_i$$

Def. The line integral of $f(x,y)$ along a plane curve C is

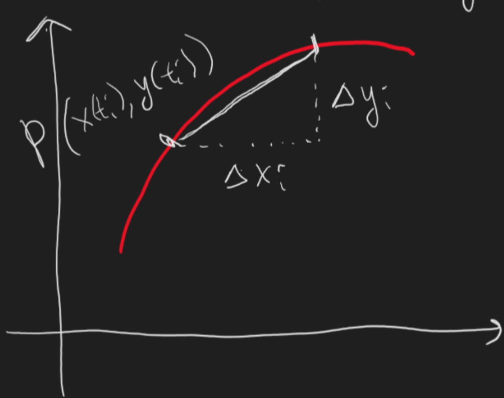
$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta s_i.$$

Def. The line integral of $f(x,y,z)$ along a space curve C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))\,\Delta s_i$$

How to compute?

$P_i(x(t_{i+1}), y(t_{i+1}))$



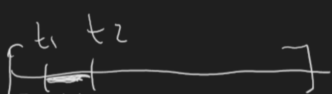
$$\Delta s_i \approx \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$\Delta x_i = x(t_{i+1}) - x(t_i) =$$

$$= \frac{x(t_i+\Delta t) - x(t_i)}{\Delta t}\cdot \Delta t$$

Used:

$t_{i+1} = t_i + \Delta t$



$$\approx x'(t_i)\cdot \Delta t$$

$$\left(\text{Because } x'(t_i) = \lim_{\Delta t\to 0} \frac{x(t_i+\Delta t) - x(t_i)}{\Delta t}\right)$$

Similarly $\Delta y_i \approx y'(t_i)\cdot \Delta t$.

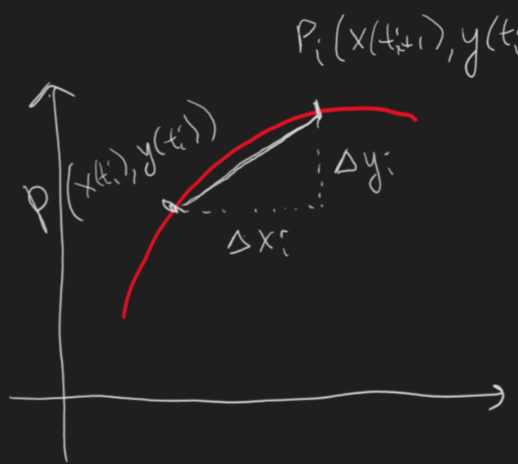
$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} \approx \sqrt{x'(t_i)^2 + y'(t_i)^2}\,\Delta t.$$

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta s_i$$

Def. The line integral of $f(x,y,z)$ along a space curve C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))\,\Delta s_i$$

How to compute?



$$\Delta s_i \approx \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

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Used:

$t_{i+1} = t_i + \Delta t$

$[\, t_1 \ t_2 \ ]$, Δt

$$\approx x'(t_i) \cdot \Delta t$$

(Because $x'(t_i) = \lim_{\Delta t \to 0} \frac{x(t_i+\Delta t) - x(t_i)}{\Delta t}$)

Similarly $\Delta y_i \approx y'(t_i) \cdot \Delta t$.

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} \approx \sqrt{x'(t_i)^2 + y'(t_i)^2}\,\Delta t.$$

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta s_i$$

$$= \lim \sum^{n} f(x(t_i), y(t_i)) \sqrt{x'(t_i)^2 + y'(t_i)^2} \cdot \Delta t =$$

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta s_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i)) \sqrt{x'(t_i)^2 + y'(t_i)^2} \cdot \Delta t =$$

$$= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \underline{\sqrt{x'(t)^2 + y'(t)^2}}\, dt}$$

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt}$$

Can combine both formulas into one.

Identify a pt $(x(t), y(t))$ [$(x(t), y(t), z(t))$] with its position vector $\bar{r}(t) = \langle x(t), y(t) \rangle$

(or $\langle x(t), y(t), z(t) \rangle$) $\qquad \bar{r}'(t) = \langle x'(t), y'(t) \rangle$

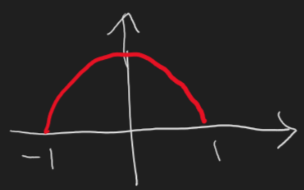
$$\boxed{\int_C f\,ds = \int_a^b f(\bar{r}(t))\, |\bar{r}'(t)|\, dt}$$

– the same formula as those above, using vector notations.

using vector notations:

Ex. Find $\int_C (2+x^2y)\,ds$, where C is the upper-half of the unit circle.

Solution Parametrization of C:



$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$.

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t \cdot \sin t)\sqrt{x'(t)^2+y'(t)^2}\,dt$$

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1}}\,dt =$$

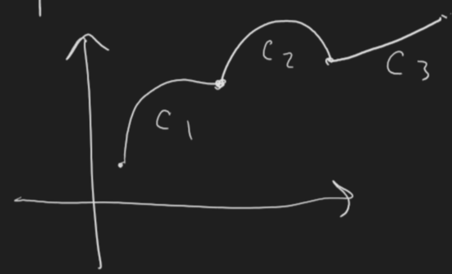
$$= \int_0^{\pi} (2+\cos^2 t \underbrace{\sin t)\,dt} = 2\pi - \int_0^{\pi} \cos^2 t\, d(\cos t) =$$

$$= 2\pi - \int_1^{-1} u^2\,du = 2\pi - \left[\frac{u^3}{3}\right]_{u=1}^{u=-1} =$$

$$= 2\pi + \frac{1}{3} + \frac{1}{3} = 2\pi + \frac{2}{3}. \quad \square$$

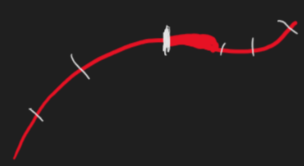
Remark We got our formulas in assumption that C is smooth. However we can use the formulas for ... the curves also.

Remark We got our formula in assumption that C is smooth. However we can use the formulas for piecewise-smooth curves also.



$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots$$

Physical interpretation: A thin wire is shaped as a curve C and density $\rho(x,y,z)$. Then the mass of the wire is $\displaystyle\int_C \rho(x,y,z)\,ds$.

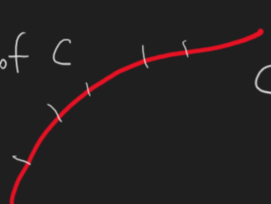


$$m = \sum \Delta m_i$$
$$= \lim \sum \rho(x(t_i), y(t_i))\,\Delta s_i$$
$$= \int_C \rho(x,y,z)\,ds.$$

Another application of line integrals:

Length of curve

L = length of C



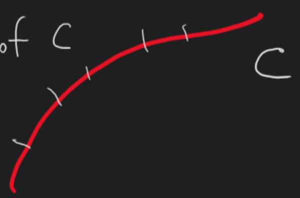
$$\int_C 1\,ds$$

$$= \int_C f(x,y,z)\, ds.$$

Another application of line integrals:

Length of curve

$L =$ length of C



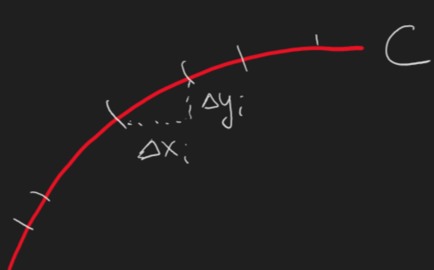
$$\int_C 1\, ds$$

$$= \lim \sum 1 \cdot \Delta s_i = \lim \sum \Delta s_i = L.$$

$$\boxed{\begin{array}{l}\text{The length of the curve } C \text{ given} \\ \text{by parametrization } \bar{r}(t) \text{ is} \\ L = \int_C 1\, ds = \int_a^b |\bar{r}'(t)|\, dt\end{array}}$$

Line integrals are also called line integrals w.r.t. the arc length.

Line integrals w.r.t. x and w.r.t. y (and w.r.t. z)



$$\lim \sum f(x(t_i), y(t_i))\, \underline{\Delta s_i}.$$

Def. The line integral of f along the curve C w.r.t. x and w.r.t. y

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$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta x_i$$

and

$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i), y(t_i))\,\Delta y_i$$

(If C is a space curve, there is also $\int_C f(x,y,z)\,dz = \ldots$).

How to compute?

Already noticed: $\Delta x_i \approx x'(t_i)\cdot \Delta t$.

$$\boxed{\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt}$$

$$\boxed{\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt}$$

It often happens that one considers

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$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

abbreviation

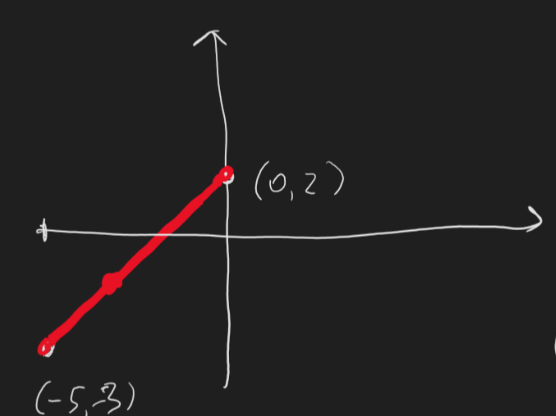
$$\int_C P(x,y)\,dx + Q(x,y)\,dy.$$

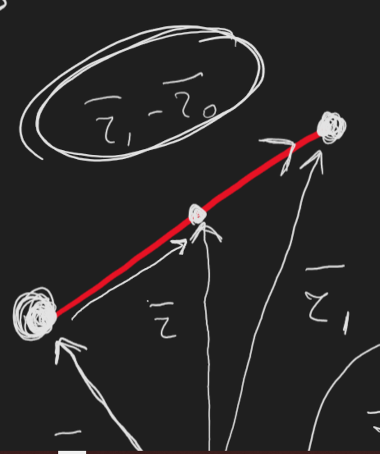
Ex. Find $\int_C y^2\,dx + x\,dy$, where

a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$.

b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Solution a) Parametrization of C:

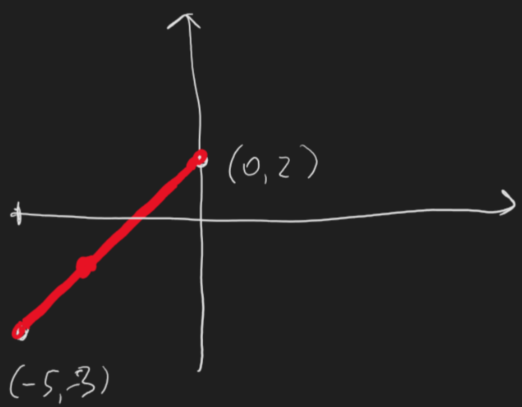


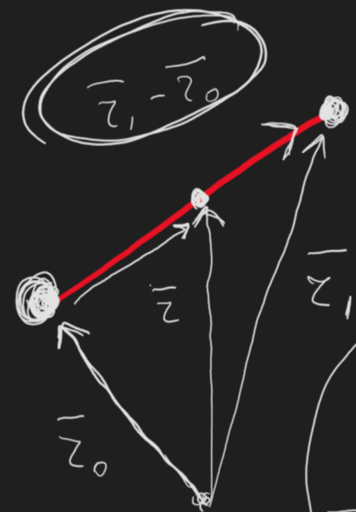


$$\bar{r} = \bar{r}_0 + t(\bar{r}_1 - \bar{r}_0)$$

$x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Solution a) Parametrization of C:





$$\boxed{\begin{aligned}\bar{r} &= \bar{r}_0 + t(\bar{r}_1 - \bar{r}_0)\\ &= (1-t)\bar{r}_0 + t\bar{r}_1\\ &\quad 0 \le t \le 1\end{aligned}}$$

$$\bar{r}(t) = (1-t)\langle -5,-3\rangle + t\langle 0,2\rangle =$$

$$= \langle -5(1-t), -3(1-t)+2t\rangle = \underline{\langle 5t-5, 5t-3\rangle}$$

is a parametrization of C. Now compute integral

$$\int_C y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 \cdot x'(t)\,dt + (5t-5)\,y'(t)\,dt$$

$$= \int_0^1 (5t-3)^2 \cdot 5\,dt + (5t-5)\cdot 5\,dt =$$

$$= 5\int_0^1 (25t^2 - 30t + 9 + 5t - 5)\,dt =$$

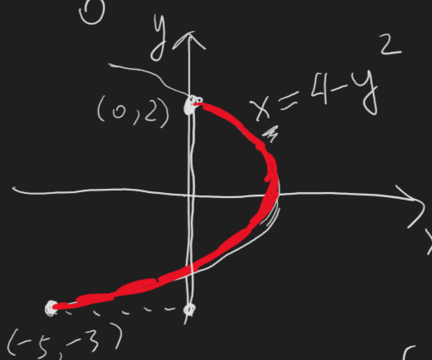
C $\quad {}^{1}_{0}$

$$= \int_0^1 (5t-3)^2 \cdot 5\,dt + (5t-5)\cdot 5\,dt =$$

$$= 5\int_0^1 (25t^2 - 30t + 9 + 5t - 5)\,dt =$$

$$= 5\int_0^1 (25t^2 - 25t + 4)\,dt = \dots$$

b)



Parametrization of C:

$\bar{r}(y) = \langle 4-y^2, y \rangle$ $\qquad -3 \le y \le 2.$

$\bar{r}(t) = \langle 4-t^2, t \rangle.$ $\qquad -3 \le t \le 2.$

$$\int_C y^2\,dx + x\,dy = \int_{-3}^{2} t^2 x'(t)\,dt + (4-t^2)\,y'(t)\,dt =$$

$$= \int_{-3}^{2} t^2(-2t)\,dt + (4-t^2)\,dt =$$

$$= \int_{-3}^{2} (-2t^2 + 4 - t^2)\,dt = \dots$$

$\square$