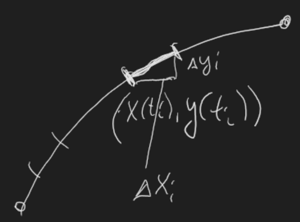
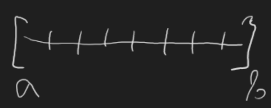
Previous lecture:

Line integral of f along C:

$$\int_C f(x,y)\,ds \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i),y(t_i))\,\boxed{\Delta s_i}$$



$\bar{r}(t) = \langle x(t), y(t)\rangle$



$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$$\int_C f\,ds = \int_a^b f(\bar{r}(t))\,|\bar{r}'(t)|\,dt.$$

Line integrals w.r.t. x and y (and z).

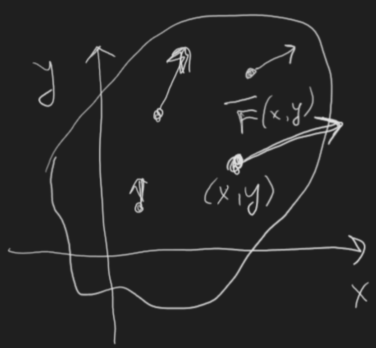
$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i),y(t_i))\,\Delta x_i$$

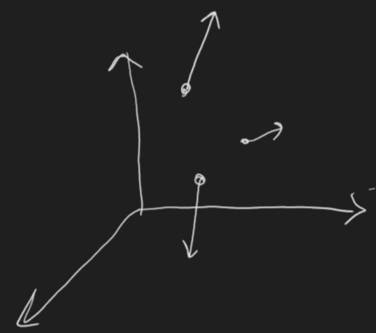
$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt.$$

...

Vector fields

Vector fields





Def. A vector field $\bar F$ on $\mathbb{R}^2$ is a function which assigns to each pt (x,y) in some region D a 2-dimensional vector $\bar F(x,y)$.

Def. A vector field $\bar F$ on $\mathbb{R}^3$ is a function which assigns to each pt (x,y,z) in some region D a 3-dimensional vector $\bar F(x,y,z)$.

We can write $\bar F(x,y) = \langle P(x,y),\ Q(x,y)\rangle$
$= P(x,y)\,\bar i + Q(x,y)\,\bar j$.

For short $\bar F = \langle P, Q\rangle = P\bar i + Q\bar j$.

Similarly for 3 variables: $\bar F = \langle P, Q, R\rangle =$
$= P\bar i + Q\bar j + R\bar k$.

Also we write $\bar F(x,y,z) = \bar F(\bar r)$

Also we write $\bar{F}(x,y,z) = \bar{F}(\bar{r})$

the p[illegible] vector of pt $(x,y,$[illegible]

Examples of vector fields:

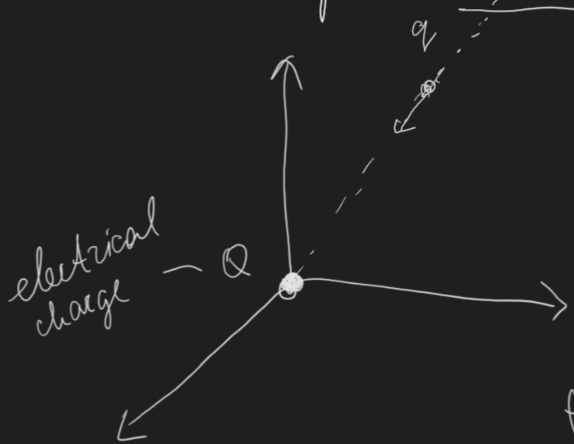
1) Velocity vector fields.

For example, a fluid is flowing and we consider the velocity vector at each pt.

2) Force vector fields.

At each pt one considers a force vector.

For example, electrical field.



If we have a charge q at $\bar{r} = \langle x, y, z \rangle$, then the electrical force acting on q is

$$\bar{F}(x,y,z) = \boxed{\frac{\varepsilon Q q}{|\bar{r}|^3}} \cdot \bar{r}$$

The electrical field $\bar{E}$ is the electrical force per unit charge.

$$\bar{E}(x,y,z) = \frac{\varepsilon Q}{|\bar{r}|^3} \cdot \bar{r}.$$

3) Gradient vector field.

Take a f-n f of several variables.

Then ∇f is a vector field.

If $f(x,y)$, then $\nabla f = \langle f_x, f_y \rangle$

$f(x,y,z)$, then $\nabla f = \langle f_x, f_y, f_z \rangle$.

$\bar{F}$ is <u>conservative</u> if $\bar{F} = \nabla f$, for some function f. In this case, f is called the <u>potential</u> f-n of $\bar{F}$.

Today will learn how to integrate vector fields.

Why one needs to integrate them?

Motivation: work done by a force.

$\bar{D}$ - displacement vector.
(An object is moved from A to B).

$\bar{F} = \text{const}$ is a force.

work done for a $\bar{F}$ $= |\bar{F}| \cos\alpha \cdot |\bar{D}| = \bar{F} \cdot \bar{D}$

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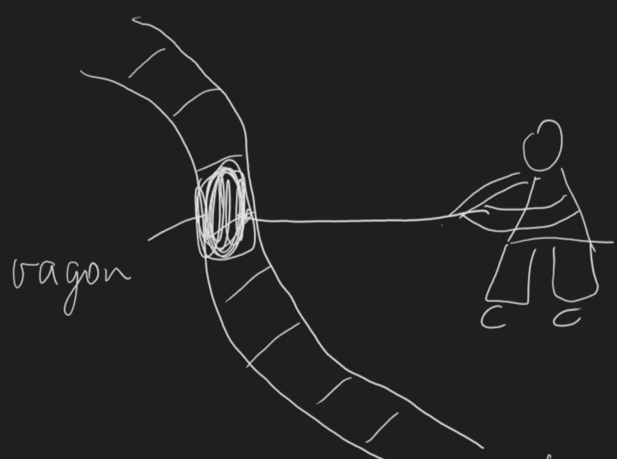
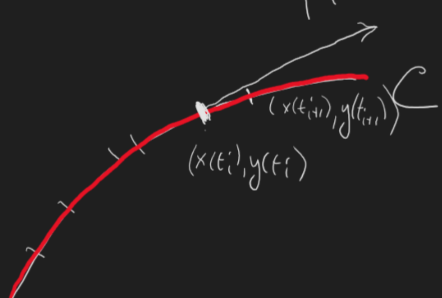
How to compute the work done if $F \neq \text{const}$ and the object is moved along a curve?



$\bar{T}(t_i)$ - the unit tangent vector.



At each little subarc
$F \approx \text{const}$,
subarc $\approx$ straight.

work done while the object is moved along subarc of length Δs_i $\approx \bar{F}(x(t_i), y(t_i)) \bullet \boxed{\Delta s_i \, \bar{T}(t_i)}$

$$\text{work done} = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\bar{F}(x(t_i), y(t_i)) \bullet \overset{\bar{T}(x(t_i), y(t_i))}{\overline{T}(t_i)}}_{f(x(t_i),\, y(t_i))} \Delta s_i$$

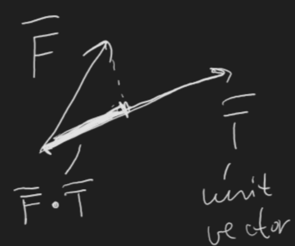
$$= \int_C \bar{F}(x,y) \bullet \bar{T}(x,y) \, ds \overset{\text{for short}}{=} \boxed{\int_C \bar{F} \bullet \bar{T} \, ds.}$$

Work done by a force F while an objec is moved along the curve C $= \int_C \bar{F} \bullet \bar{T} \, ds$

Def. The line integral of a vector field F along a curve C is

$$\int_C \bar{F} \bullet d\bar{r} = \int_C \bar{F} \bullet \bar{T} \, ds.$$

Remark $\bar{F} \bullet \bar{T}$ is the projection of the vector field onto the unit tangent vector.



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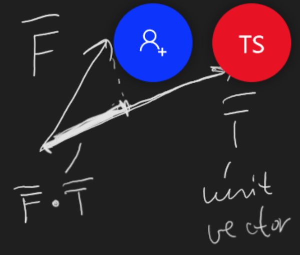
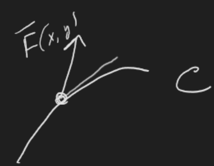


Remark $\bar F\cdot\bar T$ is the projection of the vector field onto the unit tangent vector.



How to compute $\int_C \bar F\cdot d\bar r$?

Suppose C is given by a parametrization $\bar r(t)=\langle x(t),y(t),z(t)\rangle$, $a\le t\le b$.

$$\boxed{\bar T(t)=\frac{\bar r'(t)}{|\bar r'(t)|}}$$

$$\int_C f\,ds=\int_a^b f(\bar r(t))|\bar r'(t)|\,dt$$

$$\int_C \bar F\cdot d\bar r=\int_C \bar F\cdot\bar T\,ds=\int_a^b(\bar F\cdot\bar T)(\bar r(t))\cdot|\bar r'(t)|\,dt$$

$$=\int_a^b \bar F(\bar r(t))\cdot\frac{\bar r'(t)}{\cancel{|\bar r'(t)|}}\cdot\cancel{|\bar r'(t)|}\,dt=\int_a^b \bar F(\bar r(t))\cdot\bar r'(t)\,dt$$

$$\boxed{\int_C \bar F\cdot d\bar r=\int_a^b \bar F(\bar r(t))\cdot\bar r'(t)\,dt}$$

Will write out the dot-product above in terms of coordinates and will get even nicer formula. Suppose $\bar F=\langle P,Q\rangle$

$$\int_C \bar F\cdot d\bar r=\int_a^b \bar F(\bar r(t))\cdot\bar r'(t)\,dt=$$

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt}$$

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$$\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt =$$

$$= \int_a^b \langle P(x(t),y(t)),\, Q(x(t),y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\,dt$$

$$= \int_a^b \big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\big)\,dt$$

$$= \int_C P\,dx + Q\,dy.$$

If $\bar{F} = \langle P, Q\rangle = P\bar{i} + Q\bar{j}$, then

$$\int_C \bar{F}\cdot d\bar{r} = \int_C P\,dx + Q\,dy$$

In case of 3 variables

$$\int_C \bar{F}\cdot d\bar{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Ex. Find $\int_C \bar{F} \cdot d\bar{r}$, where

$\bar{F}(x,y,z) = xy\,\bar{i} + yz\,\bar{j} + zx\,\bar{k}$ and C is given by parametric equations

$\boxed{x=t}, \; y = t^2, \; z = t^3, \quad \boxed{0 \le t \le 1}$

Solution $\int_C \bar{F} \cdot d\bar{r} = \int_C xy\,dx + yz\,dy + zx\,dz$

$$= \int_0^1 x(t)\,y(t)\,x'(t)\,dt + y(t)\,z(t)\,y'(t)\,dt + z(t)\,x(t)\,z'(t)\,dt$$

$$= \int_0^1 t \cdot t^2\,dt + t^2 \cdot t^3 \cdot 2t\,dt + t^3 \cdot t \cdot 3t^2\,dt =$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\,dt = \int_0^1 (t^3 + 5t^6)\,dt$$

$$= \ldots \qquad \square.$$

Ex. Find the work done by the force $\bar{F}(x,y) = x^2\,\bar{i} - xy\,\bar{j}$ in moving a particle along the quarter-circle $\bar{r}(t) = \cos t\,\bar{i} + \sin t\,\bar{j}, \quad 0 \le t \le \frac{\pi}{2}$.

Solution the work $= \int_C \bar{F} \cdot d\bar{r}$

$\bar r(t) = \cos t\, \bar i + \sin t\, \bar j, \quad 0 \le t \le \frac{\pi}{2}$

Solution the work $= \int_C \bar F \cdot d\bar r$

$$= \int_C x^2\, dx - xy\, dy$$

$$= \int_0^{\frac{\pi}{2}} x(t)^2 \left(x'(t)\right) dt - x(t)y(t)\, y'(t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos^2 t \cdot (-\sin t) - \cos t \sin t \cos t\right) dt$$

$$= 2\int_0^{\frac{\pi}{2}} \underbrace{\cos^2 t\left(-\sin t\right) dt} = 2\int_0^{\frac{\pi}{2}} \cos^2 t\, d(\underbrace{\cos t}_{u})$$

$$= 2\int_1^0 u^2\, du = \left[2\, \frac{u^3}{3}\right]_{u=1}^{u=0} = -\frac{2}{3}$$

□

3 Main Things to remember from today:

1) $\int_C \bar F \cdot d\bar r = \int_C \bar F \cdot \bar T\, ds$

2) work done $= \int_C \bar F \cdot d\bar r$

3) If $\bar F = \langle P, Q, R \rangle$, then

$$\int \bar F \cdot d\bar r = \int P\, dx + Q\, dy + R\, dz.$$

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$$= 2\int_1 u^2 \, du = \left[2 \frac{u^3}{3} \right]_{u=1} = -\frac{2}{3}$$

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2) work done $= \int_C \bar{F} \cdot d\bar{r}$

3) If $\bar{F} = \langle P, Q, R \rangle$, then

$$\int_C \bar{F} \cdot d\bar{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

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