Sect. 15.9: 28 Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Show that

$$\iint_R f(\underbrace{x+y}_{u})\, dA = \int_0^1 u\, f(u)\, du.$$

Change of variables: $x = x(u,v)$, $y = y(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

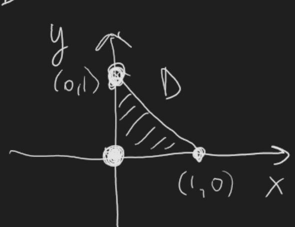
$$\iint_D f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$$

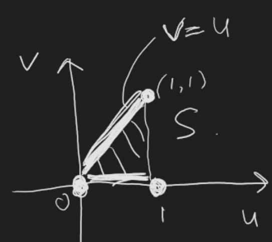
Solution: $\boxed{u = x+y, \quad v = y}$ (could introduce v in many other ways).

$x = u - v$, $y = v$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1.$$

$$\iint_D f(x+y)\, dA = \iint_S f(u)\, du\, dv$$



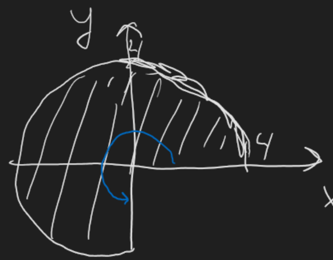


$$= \int_0^1 \int_0^u f(u)\, dv\, du = \int_0^1 f(u) [v]_{v=0}^{v=u}\, du =$$

$$= \int_0^1 f(u)\, u\, du. \qquad \square$$

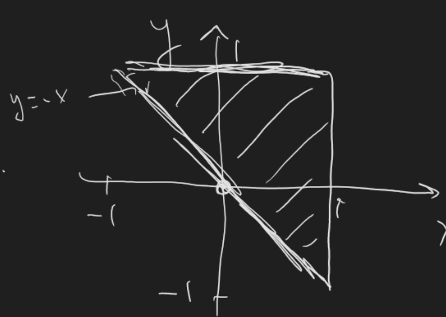
section 15.3.: 1–4, 17, 29.

1–4. R is shown. Decide whether to use polar coordinates and write $\iint_R f(x,y)\,dA$ as an iterated integral, where f is an arbitrary contin. f-n.

1. 

$R = \{(r,\theta) \mid 0 \le \theta \le \frac{3\pi}{2},\ 0 \le r \le 4\}.$

$$\iint_R f(x,y)\,dA = \int_0^{\frac{3\pi}{2}}\int_0^{4} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

2. 

$R = \{(x,y) \mid -x \le y \le 1,\ -1 \le x \le 1\}$

$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{-x}^{1} f(x,y)\,dy\,dx.$$

In polar coordinates

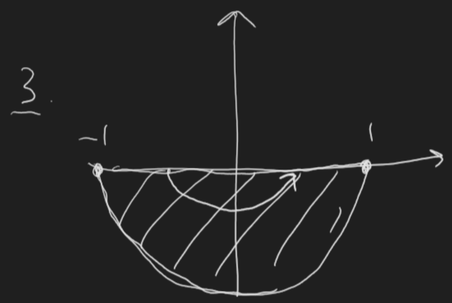
$R = \{(r,\theta) \mid -r\cos\theta \le r\sin\theta \le 1,\ -1 \le r\cos\theta \le 1\}.$

- neither type I or II.
Not clear how to draw.

2

$R = \{(r,\theta) \mid \pi \le \theta \le 2\pi,$

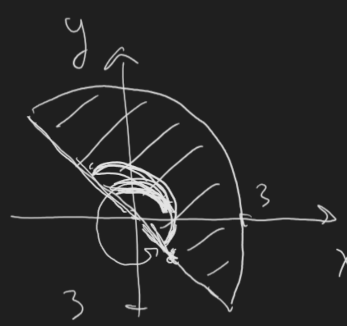
3.



$$R=\{(r,\theta)\mid \pi\le\theta\le 2\pi,\ 0\le r\le 1\}.$$

$$\iint_R f(x,y)\,dA=\int_{\pi}^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

4.



$$R=\{(r,\theta)\mid 0\le\theta\le\tfrac{3\pi}{4} \text{ and } \tfrac{7\pi}{4}\le\theta\le 2\pi,\ 0\le r\le 3\}=$$

alternatively $\{(r,\theta)\mid -\frac{\pi}{4}\le\theta\le\frac{3\pi}{4},\ 0\le r\le 3\}.$

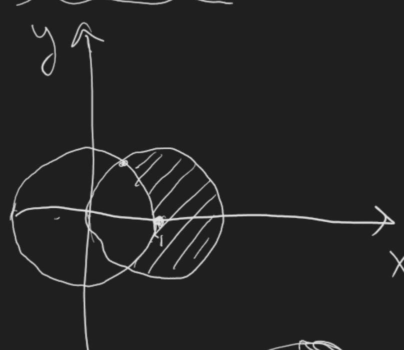
$$\iint_D f(x,y)\,dA=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

□

17. Use a double integral to find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

Solution

$$\text{area}=\iint_D dA$$

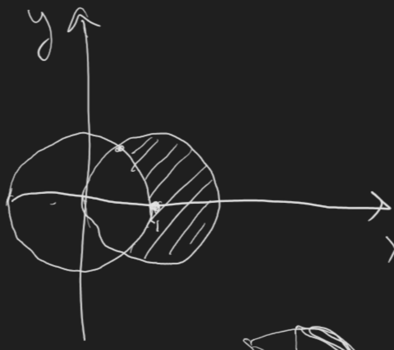


$$D=\{(x,y)\mid \underbrace{(x-1)^2+y^2\le 1}_{x^2-2x+\cancel{1}+y^2\le\cancel{1}},\ \underline{x^2+y^2\ge 1}\}$$

If we used (x,y)-coordinates, we would have to separate D into 3 pieces of type I/type II.

Solution

$$\text{area} = \iint_D dA$$



$$D=\{(x,y) \mid \underbrace{(x-1)^2+y^2\le 1}_{x^2-2x+\cancel{1}+y^2\le \cancel{1}},\ \underbrace{x^2+y^2\ge 1}\}$$

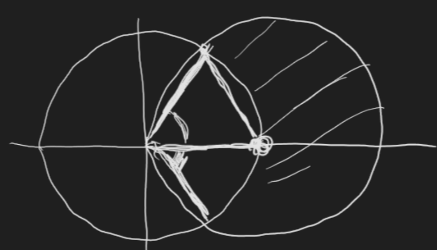
If we used (x,y) - coordinates, we would have to separate D into 3 pieces of type I / type II.

Better to use polar coordinates.

$$D=\{(r,\theta)\mid r^2-2r\cos\theta\le 0,\ r^2\ge 1\} =$$

$$=\{(r,\theta)\mid r\le \underline{2\cos\theta},\ r\ge 1\}=$$

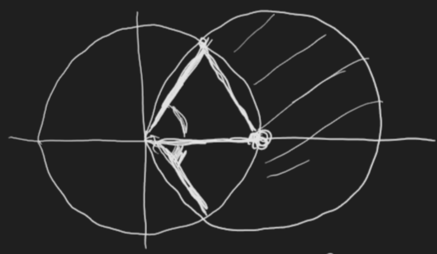
$$=\{(r,\theta)\mid 1\le r\le 2\cos\theta,\ -\frac{\pi}{3}\le\theta\le\frac{\pi}{3}\}$$



$$\iint_D dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left[\frac{r^2}{2}\right]_{r=1}^{r=2\cos\theta} d\theta=$$

$$=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(\frac{4\cos^2\theta}{2}-\frac12\right)d\theta = -\frac{\pi}{3}+\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos^2\theta\,d\theta=$$

$$=-\frac{\pi}{3}+\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(1+\cos2\theta)\,d\theta=-\frac{\pi}{3}+\frac{2\pi}{3}+\int_{-\frac{\pi}{3}}\cos2\theta\,d\theta$$



$$\iint_D dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1}^{2\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left[\frac{r^2}{2}\right]_{r=1}^{r=2\cos\theta} d\theta =$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(\frac{4\cos^2\theta}{2} - \frac{1}{2}\right)d\theta = -\frac{\pi}{3} + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos^2\theta\,d\theta =$$

$$= -\frac{\pi}{3} + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(1+\cos 2\theta)\,d\theta = -\frac{\pi}{3} + \frac{2\pi}{3} + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos 2\theta\,d\theta$$

$$= \frac{\pi}{3} + \left[\frac{1}{2}\sin 2\theta\right]_{\theta=-\frac{\pi}{3}}^{\theta=\frac{\pi}{3}} = \frac{\pi}{3} + \frac{1}{2}\left(\sin\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}. \qquad \square$$

29. $$\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx.$$

Evaluate the iterated integral by converting to polar coordinates.

Solution

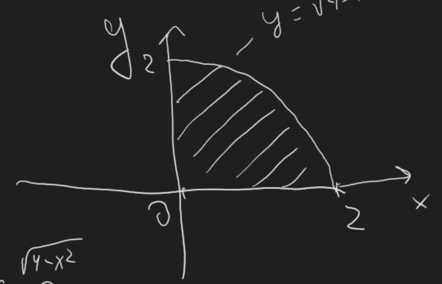
$$y = \sqrt{4-x^2}$$
$$y^2 = 4 - x^2$$
$$x^2 + y^2 = 4.$$

29. $\int_0 \int_0 e^{-x^2-y^2}\,dy\,dx.$

Evaluate the iterated integral by converting to polar coordinates.

Solution

$$y=\sqrt{4-x^2}$$
$$y^2=4-x^2$$
$$x^2+y^2=4.$$



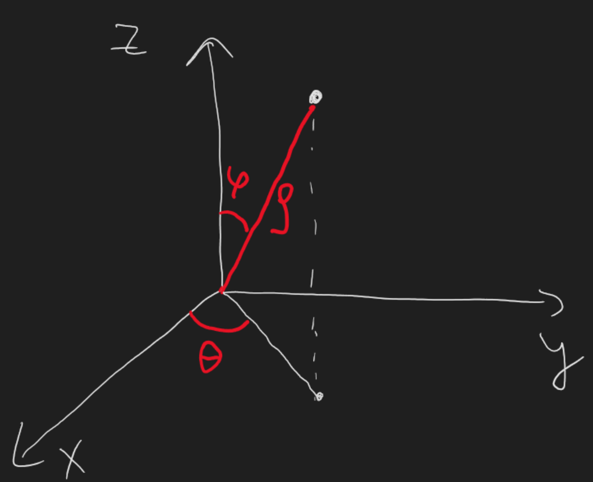
$$\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx = \iint_D e^{-x^2-y^2}\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 e^{-r^2}\cdot r\,dr\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2 \frac{1}{2}e^{-r^2}\,dr^2\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^4 \frac{1}{2}e^{-t}\,dt\,d\theta = \int_0^{\frac{\pi}{2}}\left[-\frac{1}{2}e^{-t}\right]_{t=0}^{t=4} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \underbrace{\left(-\frac{1}{2}e^{-4}+\frac{1}{2}\right)} d\theta = \frac{1}{2}\left(1-e^{-4}\right)\int_0^{\frac{\pi}{2}} d\theta =$$

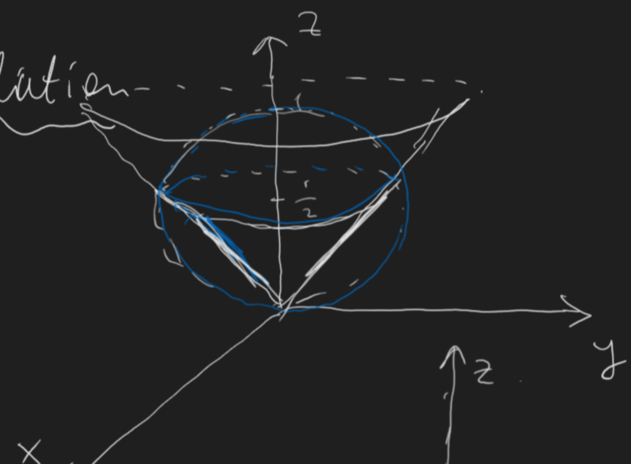
$$= \frac{\pi}{4}\left(1-e^{-4}\right). \quad \square$$

section 15.8: 15, 27.

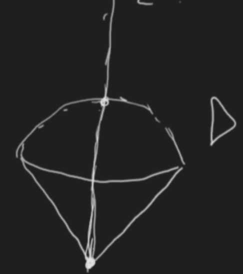


15 A solid lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$. Write a description of the solid in terms of spherical coordinates.

Solution

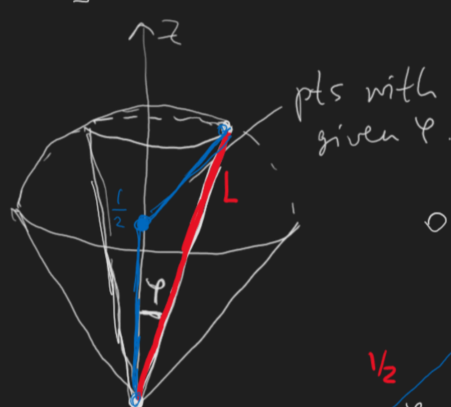


$$x^2+y^2+z^2=z$$
$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$

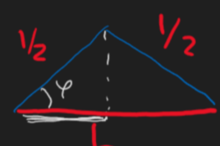


$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{4}$$

Given φ, $? \le \rho \le ?$



$$0 \le \rho \le L$$



$$L = 2\cdot\tfrac{1}{2}\cos\varphi = \cos\varphi$$

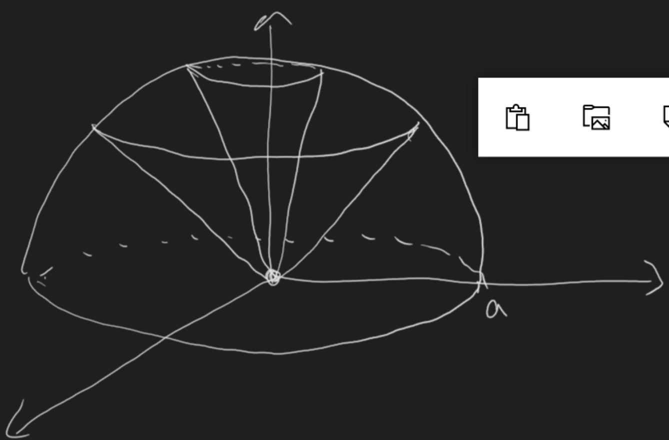
Given φ, $0 \le \rho \le \cos\varphi$.

$$D=\left\{(\rho,\varphi,\theta)\ \middle|\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{4},\ 0\le\rho\le\cos\varphi\right\}$$

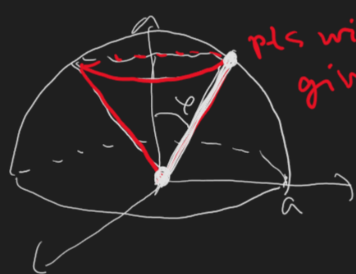
$\square$

27. Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.

Solution



$$D = \left\{ (\rho, \varphi, \theta) \;\middle|\; 0 \le \theta \le 2\pi,\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{3},\ 0 \le \rho \le a \right\}$$



pts with given φ.

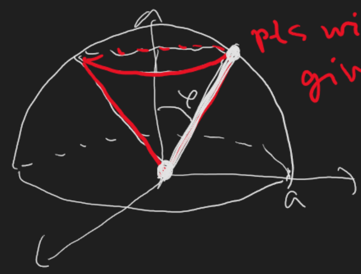
$$V = \iiint_D dV = \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^a \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \left[\sin\varphi \cdot \frac{\rho^3}{3} \right]_{\rho=0}^{\rho=a} d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \sin\varphi \cdot \frac{a^3}{3} \, d\varphi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \left[-\cos\varphi \right]_{\varphi=\frac{\pi}{6}}^{\varphi=\pi/3} d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right) d\theta = \frac{(\sqrt{3}-1)a^3}{6} \int_0^{2\pi} d\theta =$$

$$D = \left\{ (\rho, \varphi, \theta) \;\middle|\; 0 \le \theta \le 2\pi,\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{3},\ 0 \le \rho \le a \right\}$$



pts with given φ.

$$V = \iiint_D dV = \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{a} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \left[\sin\varphi \cdot \frac{\rho^3}{3} \right]_{\rho=0}^{\rho=a} d\varphi \, d\theta.$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \sin\varphi \cdot \frac{a^3}{3} \, d\varphi \, d\theta = \frac{a^3}{3}\int_0^{2\pi} \left[-\cos\varphi\right]_{\varphi=\pi/6}^{\varphi=\pi/3} d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) d\theta = \frac{(\sqrt{3}-1)a^3}{6}\int_0^{2\pi} d\theta =$$

$$= \frac{(\sqrt{3}-1)a^3}{6} \cdot 2\pi = \frac{(\sqrt{3}-1)a^3\pi}{3}. \quad \square$$