Previous lecture:

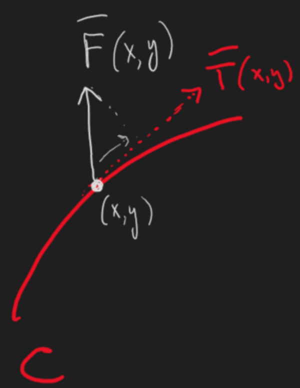
vector fields

$\bar{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ — on plane

$\bar{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ — in space.

Def.

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{T}\, ds.$$



$$\boxed{\int_C \bar{F} \cdot d\bar{r} = \int_C P\,dx + Q\,dy}$$

Work done by a force $\bar{F}$ in moving an object along C $= \int_C \bar{F} \cdot d\bar{r}$.

$\bar{F}$ is conservative if $\bar{F} = \nabla f$, for some function f.

$(\bar{F} = \langle f_x, f_y \rangle)$.

f is called a potential of $\bar{F}$.

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Conservative vector fields.

Recall: Fund. Th. of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

antiderivative

$$\left(\int_a^b F'(x)\,dx = F(b) - F(a) \right)$$

will prove an analogue for conservative vector fields.

<u>Th.</u> Suppose C is a smooth curve given by a parametrization $\bar{r}(t)$, $a \le t \le b$. Suppose f is a differentiable f-n of 2 or 3 variables. Then

$$\int_C \nabla f \cdot d\bar{r} = f(b) - f(a).$$

<u>Proof.</u> (will do for 2 variables. For 3 — the same).

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

$$\int_C \nabla f \cdot d\bar{r} = \int_C \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$$= \int_a^b \frac{\partial f}{\partial x}(\bar{r}(t))\,x'(t)\,dt + \frac{\partial f}{\partial y}(\bar{r}(t))\,y'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}(\bar{r}(t))\,\frac{dx}{dt} + \frac{\partial f}{\partial y}(\bar{r}(t))\,\frac{dy}{dt} \right) dt$$

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$$= \int_a^b \left(\frac{\partial f}{\partial x}(\bar{r}(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(\bar{r}(t)) \frac{dy}{dt} \right) dt$$

$$\underset{\text{Rule}}{\overset{\text{Chain}}{=}} \int_a^b \left(f(\bar{r}(t))\right)' \, dt \overset{\text{Fund Th. of Calc.}}{=} f(\bar{r}(b)) - f(\bar{r}(a))$$

□

Conclusion: when we compute line integrals of conservative vector fields, the curve doesn't matter! The value of the integral depends only of the initial and final pts of the curve.

Def. Suppose $\bar{F}$ is defined on region D. We say that $\int_C \bar{F} \cdot d\bar{r}$ is independent of path if

$$\int_C \bar{F} \cdot d\bar{r} = \int_{C_1} \bar{F} \cdot d\bar{r} \quad \text{for}$$

any curve C_1 in D with the same initial and final pts as C.

If $\bar{F}$ is conservative, then line integrals of $\bar{F}$ are independent of path.

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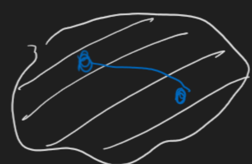
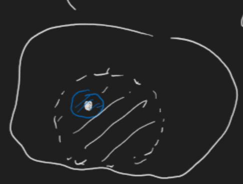
Will prove the other way around: if line integrals of $\bar{F}$ are independent of path, then $\bar{F}$ is conservative.

We say that a region D is <u>path connected</u> if any 2 pts in D can be connected by a path lying in D.

Examples: a)  path connected

b)  not path connected.

We say that D is <u>open</u> if for every pt P in D there is a disc centered in P which lies inside D. (So D doesn't contain any boundary pt).

Examples. a)  b) 

pt P in D there is a disc centered in P which lies inside D. (So D doesn't contain any boundary pts.)

Examples. a) 

$\{(x,y) \mid x^2+y^2<1\}$

open.

b) 

not open

c) 

not open

Th. Suppose $\bar{F}$ is a vector field on an open path connected region D.

Then if $\int_C \bar{F}\cdot d\bar{r}$ is independent of path, for any C inside D, then $\bar{F}$ is conservative.

Proof. (will do for 2 variables, for 3 the proof is the same).

Want: $\bar{F} \overset{?}{=} \nabla f$, for some f-n f.

(Idea from Calculus I: given g, how to find its antiderivative? antider. $(x) = \int_a^x g\,dx$

a - some fixed pt.

antiderivative of g is a f-n f s.t. $f' = g$.)

Fix (a,b). Take any curve C whose

(Idea from Calculus I: given g, how to find its antiderivative? antiderivative $\int_a^x g$

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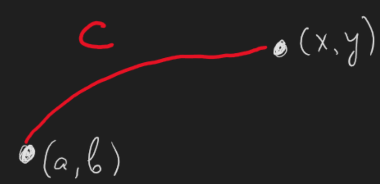
Fix (a,b). Take any curve C whose initial pt is (a,b) and final pt is (x,y)



Define $$\boxed{f(x,y) = \int_C \vec{F} \cdot d\vec{r}}$$

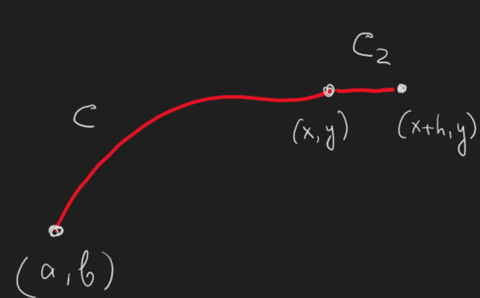
Will prove $\vec{F} \overset{?}{=} \nabla f$

$\vec{F} = \langle \underline{P}, Q \rangle$, $\nabla f = \langle \underline{f_x}, f_y \rangle$.

<u>Need to check:</u> $\boxed{f_x \overset{?}{=} P}$, $f_y \overset{?}{=} Q$.

Will check the 1^{st} one, the 2^{nd} one is similar.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

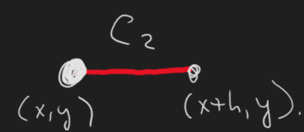


$C_1 = C \cup C_2$ - curve connecting (a,b) with $(x+h,y)$.

$$f(x+h,y) - f(x,y) = \int_{C_1} \bar{F}\cdot d\bar{r} - \int_{C} \bar{F}\cdot d\bar{r}$$

$$= \cancel{\int_{C} \bar{F}\cdot d\bar{r}} + \int_{C_2} \bar{F}\cdot d\bar{r} - \cancel{\int_{C} \bar{F}\cdot d\bar{r}}$$

$$= \int_{C_2} P\,dx + Q\,dy$$



$\bar{r}(t) = \langle x+th, y\rangle$

$0 \le t \le 1$

$$= \int_0^1 P\,x'(t)\,dt + Q\,\underbrace{y'(t)}_{=0}\,dt = \int_0^1 P\,h\,dt$$

$$= h\int_0^1 P\,dt \approx h\,P(x,y)\int_0^1 dt = h\,P(x,y).$$

$$f(x+h,y) - f(x,y) \approx h\,P(x,y).$$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{\cancel{h}\,P(x,y)}{\cancel{h}} = P(x,y).$$

Similarly $f_y = Q$.

$\nabla f = \bar{F}$. $\square$

$\bar{F}$ is conservative $\iff$

$\bar{F}$ is conservative $\Longleftrightarrow$ line integrals of $\bar{F}$ are independent of path.

Remark The assumption that D is path connected and open holds for any naturally arising examples of vector fields.

A very important question: given a vector field at hands, how to see whether it is conservative or not?

$\bar{F} = \langle P, Q \rangle$. If $\bar{F}$ is conservative,

$$\bar{F} = \nabla f.$$

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}.$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

A necessary condition for being conservative:

A necessary condition for being conservative:

If $\bar{F} = \langle P, Q \rangle$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

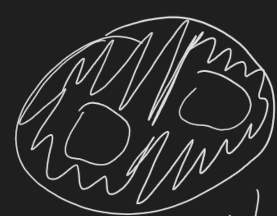
In many important cases, this condition is also sufficient for being conservative.

A region D is <u>simply connected</u> if it has no holes.

a)

simply connected

b)



not simply conn.

c) $\mathbb{R}^2 \setminus \{(0,0)\}$ not simply connected.

Will prove next time:

If $\bar{F}$ is defined on simply connected region and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\bar{F}$ is conservative

<u>Ex.</u> Is $\bar{F}$ conservative?

$$\bar{F}(x,y) = (x-y)\,\bar{i} + (x-2)\,\bar{j} = \langle x-y, x-2 \rangle$$

<u>Solution</u> $\frac{\partial P}{\partial y} \overset{?}{=} \frac{\partial Q}{\partial x}$

will prove next time:

If $\overline{F}$ is defined on simply connected region and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\overline{F}$ is conservative

Ex. Is $\overline{F}$ conservative?

$$\overline{F}(x,y) = (x-y)\,\overline{i} + (x-2)\,\overline{j} = \langle \underline{\underline{x-y}}, \underline{\underline{x-2}} \rangle.$$

Solution $\frac{\partial P}{\partial y} \overset{?}{=} \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1.$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \Rightarrow \overline{F}$ is not conservative. $\square$

Ex. $\overline{F} = \langle 3 + 2xy, x^2 - 3y^2 \rangle$. Is $\overline{F}$ conservative?

Solution $\frac{\partial P}{\partial y} \overset{?}{=} \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x.$$

$\overline{F}$ is defined on $\mathbb{R}^2$ which is simply connected. So since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we conclude $\overline{F}$ is conservative.

Ex. $\bar{F} = \langle 3+2xy,\ x^2-3y^2 \rangle$. Find

$\int_C \bar{F}\cdot d\bar{r}$, where C is the curve given by $\bar{r}(t) = \langle e^t \sin t,\ e^t \cos t \rangle$,

$0 \le t \le \pi$.

Solution Usual way is long and hard.
Instead, since $\bar{F}$ is conservative, we will try to find its potential, f, and then

$$\int_C \bar{F}\cdot d\bar{r} = f(\bar{r}(\pi)) - f(\bar{r}(0)).$$

So we want f s.t. $\bar{F} = \nabla f$.

$$f_x = 3+2xy \ \ (1), \quad f_y = x^2-3y^2 \ \ (2)$$

Integrate (1) assuming y is const:

$$\boxed{f(x,y) = 3x + x^2 y + g(y)} \quad (3)$$

To find g differentiate (3) w.r.t. y and substitute it into (2):

$$\cancel{x^2} + g'(y) = \cancel{x^2} - 3y^2$$

$$g'(y) = -3y^2$$

$$g(y) = -y^3 + C$$

$$\boxed{f(x,y) = 3x + x^2 y - y^3}$$

– a potential of $\bar{F}$.

So want f s.t.

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– a potential of $\bar{F}$.

$$\int_C \bar{F}\cdot d\bar{r} = f(\bar{r}(\pi)) - f(\bar{r}(0))$$

We have $\bar{r}(t) = \langle e^t \sin t, e^t \cos t\rangle$, $0 \le t \le \pi$.

$$\bar{r}(0) = \langle 0, 1\rangle, \qquad \bar{r}(\pi) = \langle 0, -e^{\pi}\rangle.$$

$$f(\bar{r}(\pi)) = e^{3\pi}, \qquad f(\bar{r}(0)) = -1.$$

$$\int_C \bar{F}\cdot d\bar{r} = e^{3\pi} + 1. \qquad \square$$