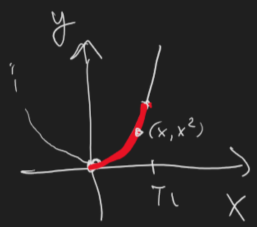
16.2 ; 5, 7, 9.

5 Evaluate $\int_C (x^2 y + \sin x)\,dy$, where C is the arc of the parabola $y=x^2$ from $(0,0)$ to (π,π^2).

Solution Parametrization of C:



$\bar r(x) = \langle x, x^2\rangle$, $0 \le x \le \pi$.

$\bar r(t) = \langle t, t^2\rangle$, $0 \le t \le \pi$.

$$\int_C (x^2 y + \sin x)\,dy = \int_0^\pi (t^2 t^2 + \sin t)\, y'(t)\,dt =$$

$$= \int_0^\pi (t^4 + \sin t)\cdot 2t\,dt = 2\int_0^\pi (t^5 + t\sin t)\,dt$$

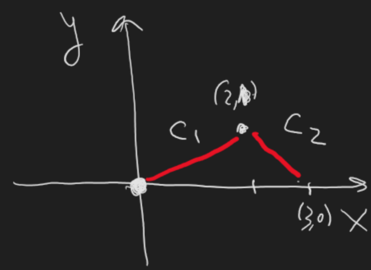
$$= \left[\frac{2t^6}{6}\right]_{t=0}^{\pi} + \int_0^\pi 2t\sin t\,dt = \frac{\pi^6}{3} - \int_0^\pi 2t\,d(\cos t) =$$

$$= \frac{\pi^6}{3} - \left[2t\cdot\cos t\right]_{t=0}^{t=\pi} + \int_0^\pi \cos t\cdot 2\,dt =$$

$$= \frac{\pi^6}{3} + 2\pi + \left[2\sin t\right]_{t=0}^{t=\pi} = \frac{\pi^6}{3} + 2\pi. \quad \square$$

7. $\int_C (x+2y)\,dx + x^2\,dy$, where C

7. $\int_C (x+2y)\,dx + x^2\,dy$, where C consists of line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$.



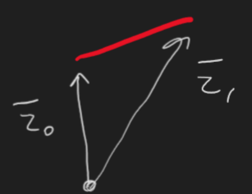
Solution

$$\int_C \dots = \int_{C_1} \dots + \int_{C_2} \dots$$

Parametrization of line segment

$$\bar z(t) = (1-t)\,\bar z_0 + t\,\bar z_1$$

$0 \le t \le 1$.



Parametrization of C_1:

$$\bar z(t) = t\langle 2, 1\rangle = \langle 2t, t\rangle \qquad 0 \le t \le 1.$$

$$\int_{C_1} (x+2y)\,\underset{\substack{\parallel \\ x'(t)dt}}{dx} + x^2\,\underset{\substack{\parallel \\ y'(t)dt \\ \parallel \\ 0}}{dy} = \int_0^1 (2t+t^2)\cdot 2\,dt = \left[2t^2+2t\right]_{t=0}^{t=1} = 4.$$

Parametrization of C_2: $\qquad 0 \le t \le 1$

$$\bar z(t) = (1-t)\langle 2,1\rangle + t\langle 3,0\rangle$$

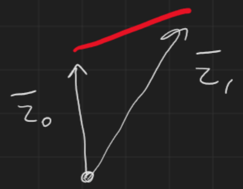
$$= \langle 2-2t+3t,\ 1-t\rangle = \langle t+2,\ 1-t\rangle$$

$$\int (x+2y)\,dx + x^2\,dy = \int^1 \left((t+2+2-2t)\,dt - (t+2)^2\,dt\right.$$

$$\int_C \qquad \int_{C_1} \qquad \int_{C_2}$$

Parametrization of line segment

$$\bar z(t) = (1-t)\,\bar z_0 + t\,\bar z_1$$
$$0 \le t \le 1.$$



Parametrization of C_1:

$$\bar r(t) = t\langle 2, 0\rangle = \langle 2t, 0\rangle \qquad 0 \le t \le 1.$$

$$\int_{C_1} (x+2y)\,\underset{x'(t)dt}{dx} + x^2\,\underset{y'(t)dt \,=\, 0}{dy} = \int_0^1 (2t+2)\cdot 2\,dt = \left[2t^2+2t\right]_{t=0}^{t=1} = 4.$$

Parametrization of C_2: $\qquad 0 \le t \le 1$

$$\bar r(t) = (1-t)\langle 2,1\rangle + t\langle 3,0\rangle$$
$$= \langle 2-2t+3t,\ 1-t\rangle = \langle t+2,\ 1-t\rangle$$

$$\int_{C_2} (x+2y)\,dx + x^2\,dy = \int_0^1 (t+2+2-2t)\,dt - (t+2)^2\,dt$$

$$= \int_0^1 \left(\cancel{4} - t - t^2 - 4t \cancel{-4}\right)dt = \int_0^1 (-t^2 - 5t)\,dt$$

$$= \left[-\frac{t^3}{3} - \frac{5t^2}{2}\right]_{t=0}^{1} = -\frac{1}{3} - \frac{5}{2} = -\frac{17}{6}$$

$$\int_C \dots = 4 - \frac{17}{6} = \frac{7}{6}. \qquad \square$$

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9. $\int_C x^2 y \, ds$, where C is given by

$x = \cos t$, $y = \sin t$, $z = t$,

$0 \le t \le \frac{\pi}{2}$.

Solution

$$\int_C x^2 y \, ds = \boxed{\int_0^{\frac{\pi}{2}} x(t)^2 \, y(t) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt}$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t \sin t \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1} \, dt =$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t \sin t \sqrt{2} \, dt = -\sqrt{2} \int_0^{\frac{\pi}{2}} \cos^2 t \, d(\underbrace{\cos t}_{u})$$

$$= -\sqrt{2} \int_1^0 u^2 \, du = \sqrt{2} \int_0^1 u^2 \, du = \sqrt{2} \left[\frac{u^3}{3} \right]_{u=0}^{u=1}$$

$$= \frac{\sqrt{2}}{3}. \quad \square$$

13.3: 1 Find the length of the curve given by parametrization

$\bar{r}(t) = \langle 2\cos t, \sqrt{5}\, t, 2\sin t \rangle$,

$-2 \le t \le 2$.

Solution $L = \int ds = \int^{2} |\bar{r}'(t)| \, dt$

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13.3:1 Find the length of the curve given by parametrization

$$\bar r(t) = \langle 2\cos t, \sqrt{5}\,t, 2\sin t\rangle,$$
$$-2 \le t \le 2.$$

Solution $L = \boxed{\int_C ds} = \boxed{\int_{-2}^{2} |\bar r'(t)|\, dt}$

$$|\bar r'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

$$= \int_{-2}^{2} \sqrt{4\sin^2 t + 5 + 4\cos^2 t}\, dt = \int_{-2}^{2} \sqrt{4+5}\, dt$$

$$= 3\int_{-2}^{2} dt = 3\cdot 4 = 12. \qquad \square$$

16.2:41 Find the work done by the force field

$$\bar F(x,y,z) = \langle x - y^2, y - z^2, z - x^2\rangle$$

on a particle that moves along the line segment from $(0,0,1)$ to $(2,1,0)$.

Solution work done by $\overbrace{\qquad}^{\text{the force field } \bar F}$ on $= \int \bar F \cdot d\bar r$

moving an object along C

16.2 : 41 Find the work done by the force field

$$\overline{F}(x,y,z) = \langle x-y^2,\ y-z^2,\ z-x^2 \rangle$$

on a particle that moves along the line segment from $(0,0,1)$ to $(2,1,0)$.

Solution

work done by $\underbrace{\qquad}_{}$ on C — the force field $\overline{F}$ — moving an object along C $= \int_C \overline{F} \cdot d\overline{r}$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

Parametrization of C:

$$\overline{r}(t) = (1-t)\langle 0,0,1\rangle + t\langle 2,1,0\rangle$$

$$= \langle 2t, t, 1-t \rangle \qquad 0 \le t \le 1.$$

$$\text{work} = \int_C (x-y^2)\underbrace{dx}_{x'(t)\,dt} + (y-z^2)\underbrace{dy}_{y'(t)\,dt} + (z-x^2)\underbrace{dz}_{z'(t)\,dt}$$

$$= \int_0^1 (2t-t^2)\,2\,dt + (t-(1-t)^2)\,dt - (1-t-4t^2)\,dt$$

$$= \int_0^1 \left(4t - 2t^2 + t - 1 + 2t - t^2 - 1 + t + 4t^2\right) dt$$

$$= \int_0^1 \left(t^2 + 8t - 2\right) dt = \left[\frac{t^3}{3} + 4t^2 - 2t\right]_{t=0}^{t=1}$$

on a particle that moves along the
line segment from $(0,0,1)$ to $(2,1,0)$

Solution: work done by the force field $\bar{F}$ on moving an object along C $= \int_C \bar{F} \bullet d\bar{r}$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

Parametrization of C :

$$\bar{r}(t) = (1-t)\langle 0,0,1\rangle + t\langle 2,1,0\rangle$$

$$= \langle 2t, t, 1-t\rangle \qquad 0 \le t \le 1.$$

$$\text{work} = \int_C (x-y^2)\,\underbrace{dx}_{x'(t)\,dt} + (y-z^2)\,\underbrace{dy}_{y'(t)\,dt} + (z-x^2)\,\underbrace{dz}_{z'(t)\,dt}$$

$$= \int_0^1 (2t-t^2)\,2\,dt + (t-(1-t)^2)\,dt - (1-t-4t^2)\,dt$$

$$= \int_0^1 \left(4t - 2t^2 + t - 1 + 2t - t^2 - 1 + t + 4t^2\right) dt$$

$$= \int_0^1 (t^2 + 8t - 2)\,dt = \left[\frac{t^3}{3} + 4t^2 - 2t\right]_{t=0}^{t=1}$$

$$= \frac{1}{3} + 4 - 2 = \frac{7}{3}. \qquad \square$$

16.2: 47 a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle'' $x^2+y^2=1$.

b) Is this also true for a force field $\bar{F}(\bar{x}) = k\bar{x}$, where k is a constant and $\bar{x} = \langle x, y\rangle$?

---

moving once uniformly around the circle:

$\bar{r}(t) = \langle \cos t, \sin t \rangle$ – standard parametr.

$0 \le t \le 2\pi$

(here pt moves once along the circle).

Twice around the circle:

$\bar{r}(t) = \langle \cos(2t), \sin(2t) \rangle$

$0 \le t \le 2\pi$

$0 \le t \le \pi$ – already make the whole circle.

$\pi \le t \le 2\pi$ – pt makes second round.

In fact we consider always only curves where pt moves once along the curve

Twice around the circle:

$$\bar r(t) = \langle \cos(2t), \sin(2t) \rangle$$

$$0 \le t \le 2\pi$$

$0 \le t \le \pi$ – already make the whole circle.

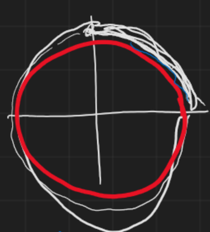
$\pi \le t \le 2\pi$ – pt makes second round.

In fact we consider always only curves where pt moves once along the curve while parameter changes.

What means to move uniformly?

Example of non-uniform moving along the circle:

$$\bar r(t) = \begin{cases} \langle \cos\frac{t}{2}, \sin\frac{t}{2} \rangle, & 0 \le t \le \pi \\ \langle \cos(\frac{3}{2}t - \pi), \sin(\frac{3}{2}t - \pi) \rangle, & \pi \le t \le 2\pi \end{cases}$$



During half-time only quoter of the circle is travelled.

During second half-time 3 quoters of the circle are travelled.

„Moving once uniformly around the circle" is another way to say that our circle is parametrized standardly.

Solution a) $\bar{F}(x,y) = \langle a, b\rangle$.

work $= \int_C \bar{F}\cdot d\bar{r} = \int_C a\,dx + b\,dy$

$\boxed{x = \cos t,\quad y = \sin t,\quad 0 \le t \le 2\pi}$

$= \int_0^{2\pi} a(-\sin t)\,dt + b\cos t\,dt$

$= \int_0^{2\pi} (b\cos t - a\sin t)\,dt$

$= \left[b\sin t + a\cos t \right]_{t=0}^{t=2\pi} = 0.$

b) $\bar{F}(\bar{x}) = k\bar{x}$ $\quad \bar{x} = \langle x, y\rangle$.

$\bar{F}(x,y) = k\langle x, y\rangle = \langle kx, ky\rangle$.

work $= \int_C \bar{F}\cdot d\bar{r} = \int_C kx\,dx + ky\,dy$

$= k\int_0^{2\pi} \cos t\,\cancel{(-\sin t)}\,dt + \cancel{\sin t}\cos t\,dt = 0.$

c) Is the same true if

$\bar{F}(x,y) = k\langle -y, x\rangle$?

$$= k\int_0 \cos t\,\cancel{(-\sin t)}\,dt + \cancel{\sin t}\,\cos t\,dt = 0.$$

c) Is the same true if

$\vec{F}(x,y) = k\langle -y, x\rangle$?

Poll: a) true

b) not true.

$$\text{work} = \int_C \vec{F}\cdot d\vec{r} = k\int_C -y\,dx + x\,dy$$

$$= k\int_0^{2\pi} (-\sin t)(-\sin t)\,dt + \cos t\cdot\cos t\,dt$$

$$= k\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= k\int_0^{2\pi} dt = 2\pi k \neq 0.$$