16.3.5 is F conservative? if yes, determine f s.t. $\nabla f = F$

$$F(x,y) = y^2 e^{xy} \vec{i} + (1+xy) e^{xy} \vec{j} = P\vec{i} + Q\vec{j}$$

recall thm 5: F conservative and P and Q have ctn 1st partial derivative

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

thm 6: P, Q ctn 1st partial derivative & D open simply connected

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\Rightarrow F \text{ conservative}$$

Here $D = \mathbb{R}^2$ open simply connected

$$\frac{\partial P}{\partial y} = 2ye^{xy} + xy^2e^{xy}$$ continuous

$$\frac{\partial Q}{\partial x} = ye^{xy} + (1+xy)ye^{xy} = 2ye^{xy} + xy^2e^{xy}$$ continuous

So $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and thm 6 gives F is conservative

• We look for f such that $\nabla f = F \Rightarrow \begin{cases} \partial_x f = P = y^2 e^{xy} & (1) \\ \partial_y f = Q = (1+xy)\, e^{xy} & (2) \end{cases}$

• integrate (2) according to y $\quad f(x,y) = \cancel{\frac{e^{xy}}{x}} + \left(ye^{xy} - \cancel{\frac{e^{xy}}{x}}\right) + g(y)$

detail: $(te^t - e^t)' = \cancel{e^t} + te^t - \cancel{e^t} = te^t$

So $f(x,y) = ye^{xy} + g(y)$

• differentiate to x $\partial_x f(x,y) = y^2 e^{xy} + g'(y) = P(x,y) = y^2 e^{xy}$

so $g'(y) = 0$ so $g(y) = cte$

So $f(x,y) = y^2 e^{xy} + cte$

and $\nabla f = F$

16.3.13 find f s.t. $\nabla f = F$ and evaluate $\int_C F \cdot dr$

$F(x,y) = x^2 y^3 \vec{i} + x^3 y^2 \vec{j}$ $\qquad C: r(t) = \langle t^3 - 2t, t^3 + 2t \rangle \qquad 0 \leq t \leq 1$

" $\partial_x f = x^2 y^3 \rightarrow f(x,y) = \dfrac{x^3 y^3}{3} + g(y) \rightarrow \partial_y f = x^3 y^2 + g'(y) = x^3 y^2$

$\rightarrow g'(y) = 0 \rightarrow g(y) = ct \rightarrow f(x,y) = \dfrac{x^3 y^3}{3} + ct$ "

and $\nabla f = F$

recall thm 2: C smooth curve: $r(t)$ $a \leq t \leq b$

f differentiable, ∇f ctn on C then

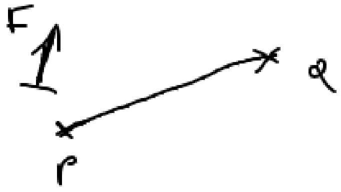
$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a)) = \frac{(1^3-2)^3(1^3+2)^3}{3} - 0 = -3^2 = -9$$

$$f(r(t)) = \frac{(t^3-2t)^3(t^3+2t)^3}{3} + c_t$$

16.3.23 Find the work done by F on a point moving from P to Q

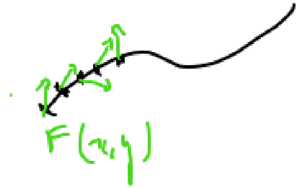
$F(x,y) = x^3\vec{i} + y^3\vec{j}$ $\quad$ P: $(1,0)$, $\quad$ Q: $(2,2)$



Recall $\quad W = "\vec{F}\cdot\vec{D}" = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C F\cdot r'(t)\,dt = \int_C F\cdot dr$

F conservative since $f(x,y) = \dfrac{x^4+y^4}{4}$ is such that

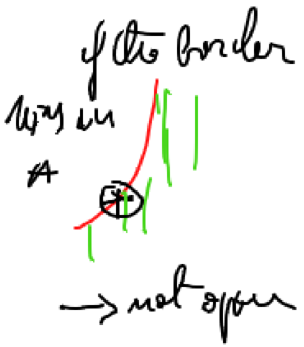
$\nabla f = F$



$$W = \int_C F\cdot dr = \int_C \nabla f\cdot dr = f(r(b)) - f(r(a)) = f(Q) - f(P) = \frac{2^4+2^2}{4} - \frac{1^4+0^4}{4} = \frac{47}{4}$$

16.3.31 a) open? b) connected? c) simply connected?

$\{(x,y) \mid y < x^2\} = A$



Recall: A is open if $\forall P \in A$, $\exists D$ disk of center P that lies in D

- "A does not contain its boundary points"

a) here $y = x^2$ is not in A so A is open

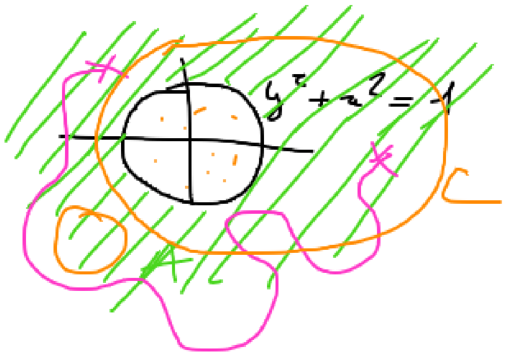
Recall: A is connected if any two points can be joined by a path that lies in D

b) here A is connected

Recall: A is simply connected if every simple closed curves in D enclose only points that are in D = "there is no hole in the domain"

c) A is simply connected

16.3.34 $\{(x,y) \mid x^2+y^2>1\} = A$



a) $x^2+y^2=1$ is not in A so A is open
(the boundary points are not in A)

b) A is connected (–)

c) (–) for the loop C, there are points enclose by C which are not in A
A is not simply connected

•  A is not connected

16.3.35

$F(x,y) = \dfrac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}$

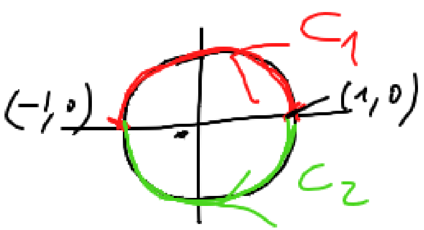
a) show that $\partial_y P = \partial_x Q$

b) show that $\int_C F \cdot dr$ is not independant of path. Does this contradict Thm 6?

a) $P(x,y) = \dfrac{-y}{x^2+y^2}$ $\qquad \partial_y P = \dfrac{-(x^2+y^2) + y\,2y}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ $\qquad$ recall $\left(\dfrac{u}{v}\right)' = \dfrac{u'v-v'u}{v^2}$

$Q(x,y) = \dfrac{x}{x^2+y^2}$ $\qquad \partial_x Q = \dfrac{(x^2+y^2) - x\,2x}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

So $\partial_y P = \partial_x Q$

b) 

- $\int_{C_1} F\cdot dr = \int_{C_1} F\cdot r'(t)\,dt = \int_0^{\pi} F(\cos t, \sin t)\cdot(-\sin t, \cos t)\,dt$

$C_1 : r(t) = (\cos t, \sin t)$

So $\int_{C_1} F\cdot dr = \int_0^{\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t}\,dt = \int_0^{\pi} 1\,dt = \pi$

- $C_2 : r(t) = (\cos(-t), \sin(-t))$ $\qquad r'(t) = (\sin(-t), -\cos(-t))$

$\int_{C_2} F\cdot dr = \int_0^{\pi} \frac{-\sin^2(-t) - \cos^2(-t)}{\cos^2(-t) + \sin^2(-t)}\,dt = -\int_0^{\pi} dt = -\pi$

So $\int_{C_1} F\cdot dr \neq \int_{C_1} F\cdot dr$

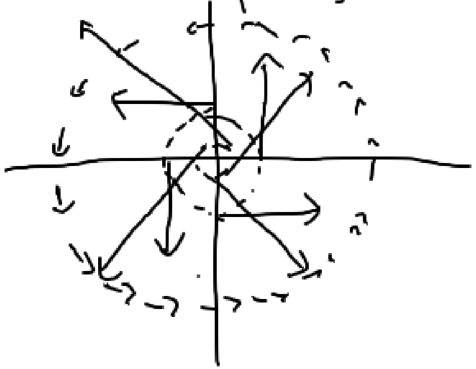
- $\mathbb{R}^2 = D$ is simply connected and open
- $\partial_y P = \partial_x Q$

<u>But</u> $\partial_y P$ and $\partial_x Q$ are not continuous

Because $\partial_y P(\frac{1}{n}, 0) = -n^2 \to -\infty$

$\partial_y P(0, \frac{1}{n}) = +n^2 \to +\infty$

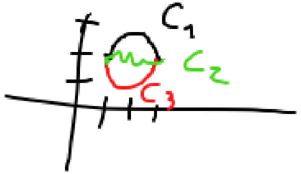
So thm 6 does not apply and there is no contradiction



$f(x,y) = \frac{---}{x^2+y^2}$

16.3.11

$F = \langle 2xy , x^2 \rangle$



a) $\int_C F \cdot dr$ is the same for C_1, C_2, C_3

We look for f such that $\nabla f = F \Rightarrow \begin{cases} \partial_x f = 2xy & (1) \\ \partial_y f = x^2 & (2) \end{cases}$

- integrate (1) regarding x $\quad f(x,y) = x^2 y + g(y)$
- differentiate regarding y $\quad f_y(x,y) = x^2 + g'(y)$
- with (2) we have $g'(y) = 0 \Rightarrow g(y) = cte$
- $f(x,y) = x^2 y + cte$

So F is conservative so $\int_C F \cdot dr$ is independent of path

so it doesn't change between C_1, C_2, C_3 since they have

the same start point and end point.

b) $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(c(b)) - f(c(a)) = f(3,2) - f(1,2)$

$= 3^2 \times 2 - 1^2 \times 2 = 16$