Previous lecture

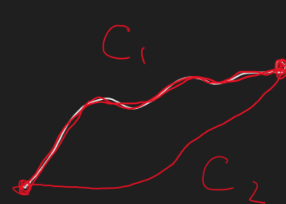
studied conservative vector fields
(means $\bar{F} = \nabla f$, for some f)
a potential.

Fund. Th. for line integrals:

$$\int_C \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a)).$$

$\bar{F}$ is conservative $\iff$ its line integrals don't depend on path.

$$\int_{C_1} \bar{F} \cdot d\bar{r} = \int_{C_2} \bar{F} \cdot d\bar{r}$$



Suppose $\bar{F} = \langle P, Q \rangle$ is defined on $D \subseteq \mathbb{R}^2$.

$\bar{F}$ is conservative $\Longrightarrow$ $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Will show today: $\Longleftarrow$ is true when D is simply-connected (has no holes).

Will deduce it from Green's Th

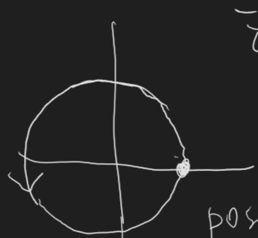
Green's Theorem

It deals with closed curves (meaning initial pt = final pt).

$a \le t \le b$

If a pt on the curve C moves counter-clockwisely when t increases, then C is positively oriented.

If clockwisely, then negatively oriented.

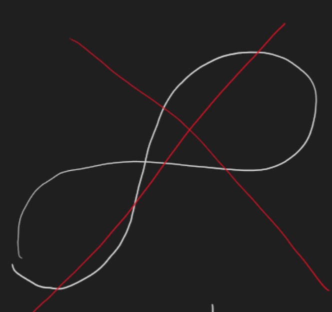


$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

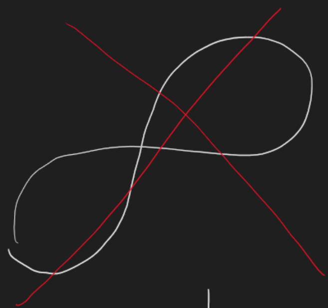
positively oriented.

negatively.



not simple

A curve is simple if it is not self-intersecting.



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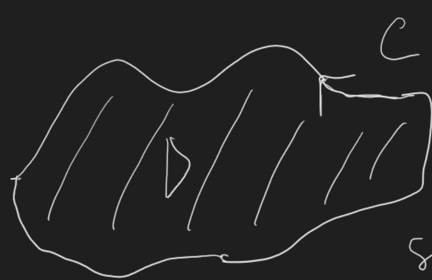


simple closed positively oriented.

Assumption for the rest of the course: all f-s are smooth (have continuous partial derivatives)

Green's Th. Suppose C is simple closed positively oriented curve and let D be the region bounded by C. Suppose P and Q are f-s of 2 variables defined on D. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Remark If C is negatively oriented, then

$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Then $\int_C \ldots = -\int_{-C} \ldots$ — same curve with opposite orientation.

Remark $\vec{F} = \langle P, Q \rangle$.

$$\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r}$$

Hence Green's Th. expresses line integrals of vector fields in terms of double integrals.

Corollary If $\vec{F} = \langle P, Q \rangle$ is defined on simply-connected region and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative.

Proof **Want:** $\vec{F}$ is conservative $\iff$ line integrals of $\vec{F}$ don't depend on path.

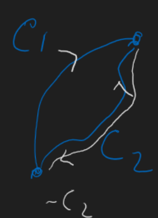
Observation: line integrals of $\vec{F}$ don't depend

Corollary If $\bar{F}$ is defined on simply-connected region and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\bar{F}$ is conservative.

Proof Want: $\bar{F}$ is conservative $\iff$ line integrals of $\bar{F}$ don't depend on path.

Observation: line integrals of $\bar{F}$ don't depend on path $\iff$ $\int_C \bar{F}\cdot d\bar{r} = 0$, for any closed curve C.

(Indeed,



$C = C_1 \cup (-C_2)$ – closed path.

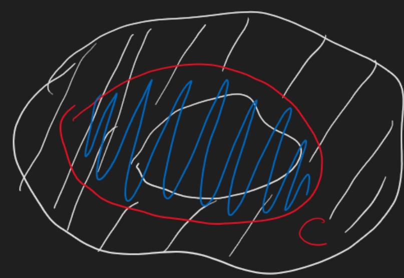
$$0 = \int_C \bar{F}\cdot d\bar{r} = \int_{C_1}\dots + \int_{-C_2} = \int_{C_1}\dots - \int_{C_2}\dots$$

$$\int_{C_1}\dots = \int_{C_2}\dots \;)$$

Want: for any closed curve C, $\int_C \bar{F}\cdot d\bar{r} = 0$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C P\,dx + Q\,dy \overset{\text{Green's th.}}{=} \iint_D \Big(\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0}\Big)\,dA = 0 \qquad \square$$

Why we needed simply connected?



Poll: It is known that $\int_C \bar{F}\cdot d\bar{r} = 5$, where C is the unit circle. Is $\bar{F}$

a) conservative

b) not conservative

c) not enough info to answer.

Because $\bar{F}$ conservative $\Rightarrow \int_C \bar{F}\cdot d\bar{r} = 0$, for any closed curve C.

Ex. Compute the work done by the force field $\bar{F} = \langle x^5, xy\rangle$ in moving a particle along the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

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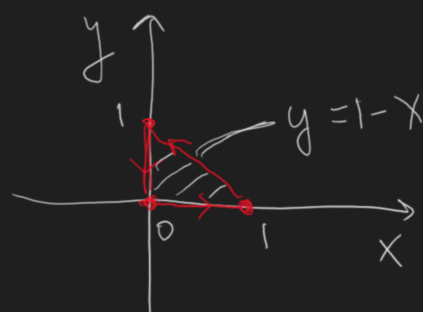
Solution



$$\text{work} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C x^5\,dx + xy\,dy$$

Usual way — easy but long
(would have to compute 3 integrals)

will use Green's Th. instead.

$$\text{work} = \int_C \underbrace{x^5}_{P}\,dx + \underbrace{xy}_{Q}\,dy \overset{\text{Green}}{=\!=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D y\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$

$$\int^1 \frac{(\overbrace{1-x}^{u})^2}{\ldots} \qquad \int^0 u^2\,du =$$

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Will use Green's Th. instead.

$$\text{work} = \int_C \underbrace{x^5}_{P}\,dx + \underbrace{xy}_{Q}\,dy \overset{\text{Green}}{=\!=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

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$$= \int_0^1 \frac{(\overset{u}{1-x})^2}{2}\,dx = -\int_1^0 \frac{u^2}{2}\,du =$$

$$= \int_0^1 \frac{u^2}{2}\,du = \left[\frac{u^3}{6}\right]_{u=0}^{u=1} = \frac{1}{6}.$$

$\square$

Ex. Compute $\int_C \underbrace{e^{x \sin x}}_{P}\,dx + \underbrace{\left(\sqrt{y^2+2} - x\right)}_{Q}dy$,

where C is the unit circle.

Poll: What is better

a) usual way

b) use Green ?

Solution $\int_C \ldots \overset{\text{Green}}{=\!=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

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Solution $\displaystyle\int_C \dots \overset{\text{Green}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$= \iint_D (-1)\, dA = -\iint_D dA = -\pi. \qquad \square$$

Ex. Compute $\displaystyle\int_C \overset{P}{x e^{x^2+1}}\, dx + \overset{Q}{(y^2-1)^2}\, dy$,

where C is the line segment from $(0,0)$ to $(1,1)$.

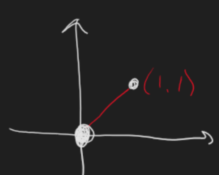
Poll: what is better

a) usual way

b) use Green ?

cannot use

Solution



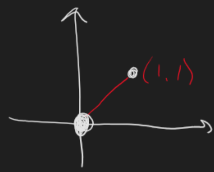
$$\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,1\rangle$$
$$= \langle t, t\rangle \qquad 0 \le t \le 1.$$

$$\int_C x e^{x^2+1}\, dx + (y^2-1)^2\, dy = \int_0^1 t e^{t^2+1}\, dt + (t^2-1)^2\, dt$$

$$= \int_0^1 t e^{\overset{u}{t^2+1}}\, dt + \int_0^1 (t^2-1)^2\, dt$$

cannot use

Solution



$\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,1\rangle$

$= \langle t,t\rangle \qquad 0\le t\le 1.$

$$\int_C x e^{x^2+1}\,dx + (y^2-1)^2\,dy = \int_0^1 t e^{t^2+1}\,dt + (t^2-1)^2\,dt$$

$$= \int_0^1 t e^{t^2+1}\,dt + \int_0^1 (t^2-1)^2\,dt$$

u

$$= \frac{1}{2}\int_1^2 e^u\,du + \int_0^1 (t^4-2t^2+1)\,dt = \dots$$

□.

Green's th. can be also used the other way around: if $\iint_D f\,dA$ is hard to compute, then try to write

$f = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, for some P, Q, and then use Green.

<u>Very important case</u>: to find area of a region bounded by a curve given by parametrization.

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a region bounded by a curve given by parametrization.

$$\text{area} = \int_D 1 \, dA$$

Could do if curve was given in terms of coordinates, but not in terms of parametrization.

Will use Green.

$$1 \stackrel{?}{=} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \quad \text{for some } P \text{ and } Q.$$

For example, take $\boxed{Q(x,y) = x}$, $\boxed{P(x,y) = 0}$. Then

$$\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.}$$

Hence

$$\underset{\text{of } D}{\text{area}} = \int_D 1 \, dA \stackrel{\text{Green}}{=\!=} \int_C P\,dx + Q\,dy = \int_C x\,dy.$$

C: curve bounding D.

$$\boxed{\text{area of } D = \int_C x\,dy}$$

Also could take $P(x,y) = -y$, $Q(x,y) = 0$.

$$\boxed{\text{area of } D = \int_C x\,dy}$$

Also could take $P(x,y) = -y$, $Q(x,y) = 0$.

Then $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$.

$$\text{area of } D = \iint_D 1\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{\text{Green}}{=} \int_C P\,dx + Q\,dy$$

$$= \int_C (-y)\, dx = -\int_C y\, dx.$$

$$\boxed{\text{area of } D = -\int_C y\, dx}$$

Add 2 formulas: $2\cdot\text{area of } D = \int_C x\,dy - y\,dx$

$$\boxed{\text{area of } D = \frac{1}{2}\int_C x\,dy - y\,dx}$$

Ex. Find the area of the region bounded by the curve given by parametrization
$\vec{r}(t) = \langle a\cos t,\ b\sin t\rangle \qquad 0 \le t \le 2\pi.$

Solution $\text{area} = \dfrac{1}{2}\displaystyle\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos t\, b\cos t\, dt$
$- b\sin t\,(-a\sin t)\,dt$

2π

Ex. Find the area of the region bounded by the curve given by parametrization
$\bar{r}(t) = \langle a\cos t, \underline{b\sin t} \rangle \quad 0 \le t \le 2\pi.$

Solution area $= \frac{1}{2} \int_C x\,dy - y\,dx = \frac{1}{2} \int_0^{2\pi} a\cos t\, b\cos t\, dt$

$- b\sin t\,(-a\sin t)\, dt$

$= \frac{1}{2} \int_0^{2\pi} ab\,(\cos^2 t + \sin^2 t)\, dt = \frac{ab}{2} \int_0^{2\pi} dt = \pi ab.$

$\square$