Previous lecture:

Green's Th.: $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

Surfaces

Curve: $\bar{r}(t) \quad a \le t \le b$

is a 1-dimensional object.

Surface is a 2-dimensional object, it is parametrized by 2 parameters.

<u>Def.</u> <u>A surface</u> is given by a vector-function

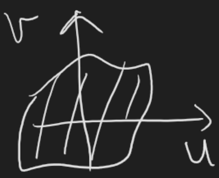
$$\bar{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$$
$$= x(u,v)\,\bar{i} + y(u,v)\,\bar{j} + z(u,v)\,\bar{k}.$$

For a curve the domain of parameter is

For a curve, the domain of parameter is an interval.

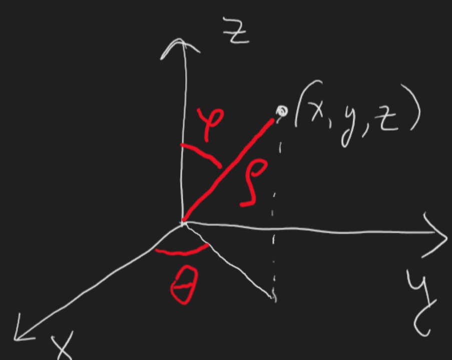
For a surface, the domain of parameters can be any region D in the uv-plane.

Most often: $\mathbb{R}^2$ or a rectangle.

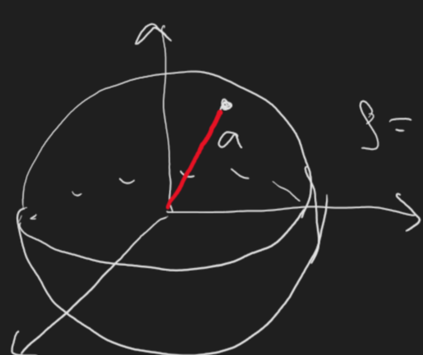


Examples:

a) Sphere of radius a centered at (0,0,0).



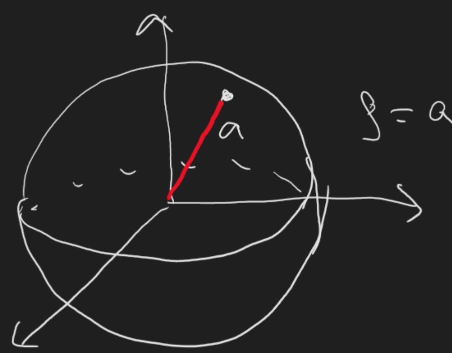
$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\theta$$



$\rho = a$

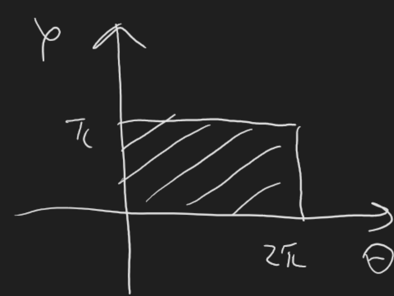
$$\vec{r}(\varphi,\theta) = \langle a\sin\varphi\cos\theta, \; a\sin\varphi\sin\theta, \; a\cos\theta\rangle$$

$x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z =$



$\bar{r}(\varphi,\theta) = \langle a \sin\varphi \cos\theta,\ a \sin\varphi \sin\theta,\ a \cos\theta \rangle$

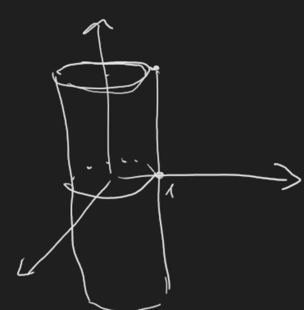
$0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$.



b) Cylinder $x^2 + y^2 = 1$

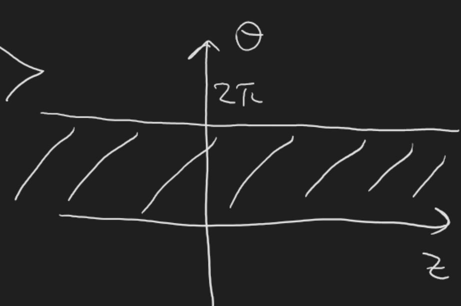
A pt on the cylinder has coordinates

$(\cos\theta,\ \sin\theta,\ z)$



$\bar{r}(\theta, z) = \langle \cos\theta,\ \sin\theta,\ z \rangle$

$0 \le \theta \le 2\pi$, $-\infty < z < \infty$



of 2 variables

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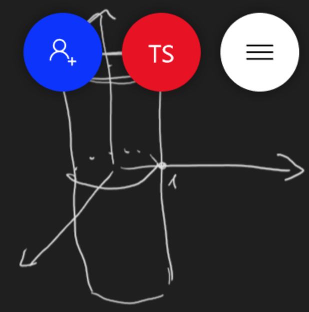

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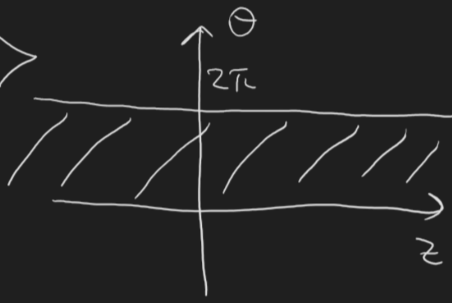
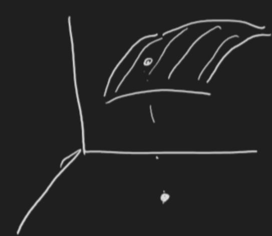


c) Graph of a f-n of 2 variables.

$z = g(x, y)$

$(x, y, g(x, y))$.



$\bar{r}(x, y) = \langle x, y, g(x, y) \rangle$.

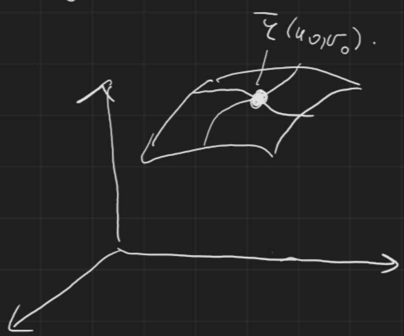
D is the domain of g.

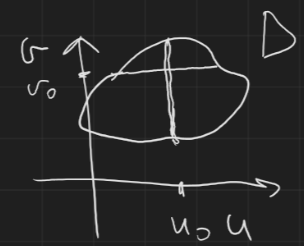
On a surface there are 2 families of useful curves called <u>grid curves</u>.

$\bar{r}(u, v)$ – parametrization.

Consider $\bar{r}(u_0, v)$ is a curve.

Fix v_0. $\bar{r}(u, v_0)$ is a curve.

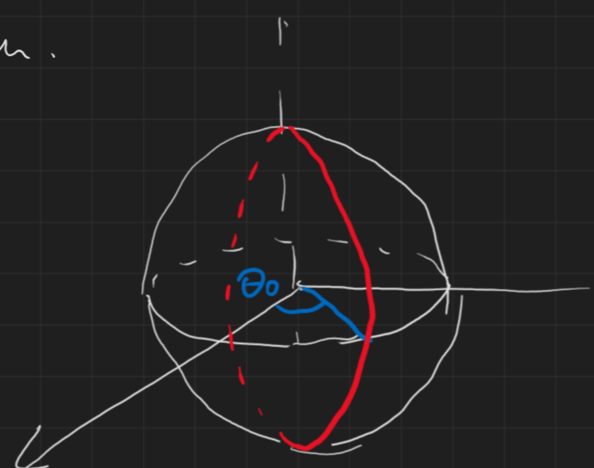




Grid curves on sphere

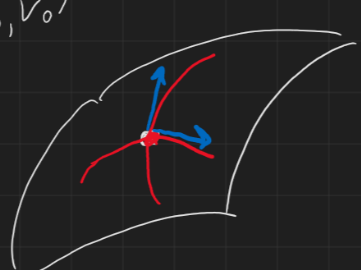
$\bar{r}(\varphi_0, \theta)$ is a parallel.

$\bar{r}(\varphi, \theta_0)$ is a meridian.



Tangent planes

$\bar{r}(u_0, v_0)$.



Tangent plane at $\bar{r}(u_0, v_0)$ contains tangent vectors to the grid curves

Tangent planes

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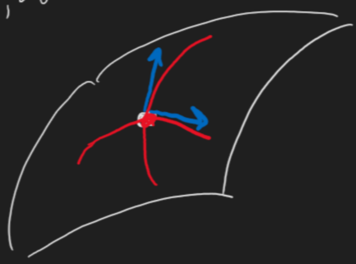

Tangent plane at $\bar{r}(u_0,v_0)$ contains tangent vectors to the grid curves $\bar{r}(u_0, v)$ and $\bar{r}(u, v_0)$.

Tangent vector to $\bar{r}(u_0, v)$ is $\bar{r}_v(u_0, v_0)$.

=//= to $\bar{r}(u, v_0)$ is $\bar{r}_u(u_0, v_0)$.

The tangent plane at $\bar{r}(u_0,v_0)$ is the plane containing the vectors $\bar{r}_v(u_0,v_0)$ and $\bar{r}_u(u_0,v_0)$.

The normal vector to the tangent plane is $\bar{r}_v(u_0,v_0) \times \bar{r}_u(u_0,v_0)$.

Ex. a) Find a normal vector to the tangent plane to the surface given by parametrization $\bar{r}(u,v) = u^2\bar{i} + v^2\bar{j} + (u+2v)\bar{k}$ at the pt corresponding to $u=1,\ v=1$.

b) write out the equation of this tangent plane.

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Ex. a) Find a normal vector to the tangent plane to the surface given by parametrization $\bar{r}(u,v) = u^2\bar{i} + v^2\bar{j} + (u+2v)\bar{k}$ at the pt corresponding to $\boxed{u=1,\ v=1.}$

b) write out the equation of this tangent plane.

Sol. $\bar{r}(u,v) = \langle u^2, v^2, u+2v \rangle$

$$\bar{r}_u(u,v) = \langle 2u, 0, 1 \rangle$$

$$\bar{r}_u(1,1) = \boxed{\langle 2, 0, 1 \rangle.}$$

$$\bar{r}_v(u,v) = \langle 0, 2v, 2 \rangle$$

$$\bar{r}_v(1,1) = \boxed{\langle 0, 2, 2 \rangle.}$$

$$\bar{r}_u(1,1) \times \bar{r}_v(1,1) = \det \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$= -2\bar{i} - 4\bar{j} + 4\bar{k} = \langle -2, -4, 4 \rangle$$

is a normal vector to the tangent plane. □

b) Recall: normal vector $\bar{n} = \langle a, b, c \rangle.$

pt on plane $(x_0, y_0, z_0).$

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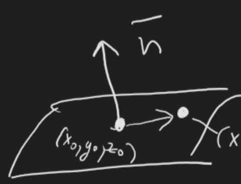


$$\langle a,b,c\rangle \cdot \langle x-x_0,\, y-y_0,\, z-z_0\rangle = 0$$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0.}$$

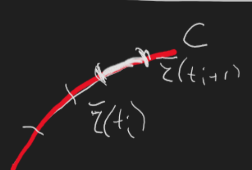
In our case $\langle a,b,c\rangle = \langle -2,-4,4\rangle$.

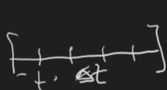
Substitute $u=1, v=1$ into $\bar r(u,v)$: $(x_0,y_0,z_0) = (1,1,3)$.

$$\boxed{-2(x-1)-4(y-1)+4(z-3)=0.}$$

Surface integrals of f-s.

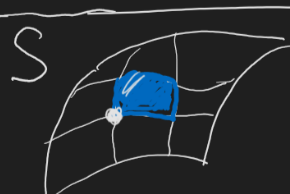
Curves

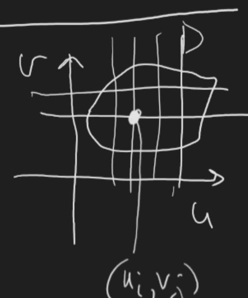




$$\int_C f\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\bar r(t_i))\,\Delta s_i$$

Surfaces



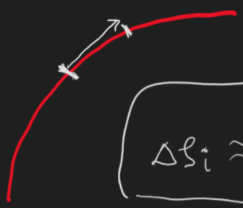


Def.

$$\iint_S f\,dS = \lim_{\substack{n\to\infty\\ m\to\infty}}\sum_{j=1}^{m}\sum_{i=1}^{n} f(\bar r(u_i,v_j))\,\Delta S_{ij}$$

$$\int_C f\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar r(t_i))\,\Delta s_i$$

$$\int_C ds = \text{length of } C.$$



$$\boxed{\Delta s_i \approx \bar r'(t_i)\,\Delta t}$$

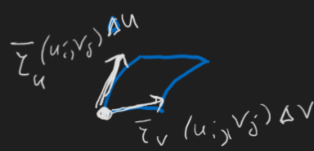
$$\int_C f\,ds = \int_a^b f(\bar r(t))\,|\bar r'(t)|\,dt$$

Def.

$$\iint_S f\,dS = \lim_{\substack{n\to\infty\\ m\to\infty}} \sum_{j=1}^{m}\sum_{i=1}^{n} f(\bar r(u_i,v_j))\,\Delta S_{ij}$$

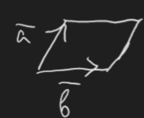
$$\iint_S dS = \text{area of } S.$$

How to compute?



$$\Delta S_{ij} = |\bar r_u(u_i,v_j) \times \bar r_v(u_i,v_j)|\,\Delta u\,\Delta v$$

(Recall:



area $= |\bar a \times \bar b|$.)

$$\iint_S f\,dS = \lim \sum\sum f(\bar r(u_i,v_j))\cdot |\bar r_u(u_i,v_j) \times \bar r_v(u_i,v_j)|\,\Delta u\,\Delta v$$

$$= \iint_D f(\bar r(u,v))\,\boxed{|\bar r_u(u,v) \times \bar r_v(u,v)|}\,dA$$

$$\boxed{\iint_S f\,dS = \iint_D f(\bar r(u,v))\,|\bar r_u \times \bar r_v|\,dA}$$

$$\boxed{\iint_S f\, dS = \iint_D f(\bar r(u,v))\,|\bar r_u \times \bar r_v|\, dA}$$

$$\boxed{\text{area of } S = \iint_D |\bar r_u \times \bar r_v|\, dA}$$

Phys. meaning of $\iint_S f\, dS$: if S is made of material of density $\rho(x,y,z)$, then $\iint_S \rho\, dS$ is the mass of S.

Ex. Find $\iint_S xz\, dS$, where S is the cone with parametric equations

$x = u\cos v$, $y = u\sin v$, $z = u$,

$0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$.

Solution $\bar r(u,v) = \langle u\cos v, u\sin v, u\rangle$.

$\bar r_u = \langle \cos v, \sin v, 1\rangle$.

Solution $\bar{r}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle.$

$$\bar{r}_u = \langle \cos v,\ \sin v,\ 1\rangle.$$

$$\bar{r}_v = \langle -u\sin v,\ u\cos v,\ 0\rangle.$$

$$\bar{r}_u \times \bar{r}_v = \det\begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{pmatrix}$$

$$= -u\cos v\,\bar{i} + u\sin v\,\bar{j} + (u\cos^2 v + u\sin^2 v)\bar{k}$$

$$= -u\cos v\,\bar{i} + u\sin v\,\bar{j} + u\bar{k}.$$

$$|\bar{r}_u \times \bar{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2}$$

$$= \sqrt{2u^2} = u\sqrt{2}$$

$$\iint_S xz\,dS = \iint_D u^2\cos v\ |\bar{r}_u \times \bar{r}_v|\,dA$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}}\int_0^1 u^3\cos v\,du\,dv = \dots \text{ easy. } \square$$

The ~~graph~~ case of graph of a f-n: $\underline{\underline{z = g(x,y)}}$

The case of graph of a f-n: $z = g(x,y)$

$$\bar{r}(x,y) = \langle x, y, g(x,y) \rangle.$$

$$\bar{r}_x = \langle 1, 0, g_x \rangle$$

$$\bar{r}_y = \langle 0, 1, g_y \rangle$$

$$\bar{r}_x \times \bar{r}_y = \det \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{pmatrix}$$

$$= -g_x \bar{i} - g_y \bar{j} + \bar{k}$$

$$|\bar{r}_x \times \bar{r}_y| = \sqrt{g_x^2 + g_y^2 + 1} \equiv \sqrt{z_x^2 + z_y^2 + 1}$$

When S is a graph $z = g(x,y)$,

$$\iint_S f\, dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + z_x^2 + z_y^2}\, dA.$$

$$\text{area of } S = \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dA.$$

Ex. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 9$.

$$\iint_S f\, dS = \iint_D f(x,y,z(x,y)) \sqrt{1+z_x^2+z_y^2}\, dA.$$

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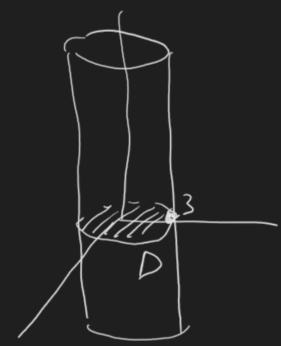

Solution Need area of graph $z=x^2+y^2$ that hangs over disc of radius 3.

$$\text{area} = \iint_D \sqrt{1+z_x^2+z_y^2}\, dA = \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

... use polar coordinates. □