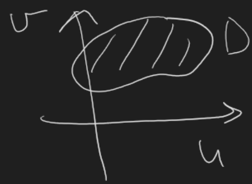
Previous lecture

Surface $\rightsquigarrow$ parametrization

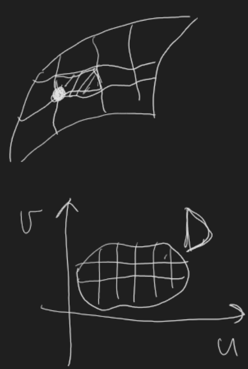
$$\bar{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$$

(u,v) is in D.



A normal vector to the surface at $\bar{r}(u,v)$: $\bar{r}_u(u,v) \times \bar{r}_v(u,v)$.

Surface integrals of f-s:



$$\boxed{\iint_S f(x,y,z)\,dS = \lim \sum_{i,j} f(r(u_i,v_j))\,\Delta S_{ij}}$$

Formula for computation:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\bar{r}(u,v))\,|\bar{r}_u \times \bar{r}_v|\,dA$$

$$\text{Area of } S = \iint_S 1\,dS.$$

S is graph of f-n a

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$$\text{Area of } S = \iint_S 1 \, dS.$$

In particular case when S is graph of f-n g

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,z(x,y)) \sqrt{1+z_x^2+z_y^2}\, dA.$$

Surface integrals of vector fields.

Why one needs to integrate vector fields over surfaces:

Over curves:

work done by a force field in moving a particle along C $= \int_C \bar{F} \cdot d\bar{r}$.

Over surfaces: consider a velocity field $\bar{F}$ of some fluid of density, say, 1.

The rate of flow through S, that is how much liquid goes through S per unit time is expressed by $\iint_S \bar{F} \cdot d\bar{S}$.

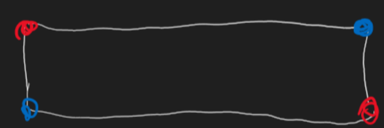
Same for heat flow, for electrical field ...

Will consider only „usual", 2-sided, surfaces.

(called orientable)

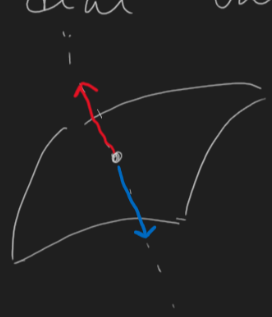
Möbius strip.

$\rightsquigarrow$ cylinder



$\rightsquigarrow$ Möbius strip.
(1-sided, or
non-orientable, surface).

Will deal only with orientable surfaces.



At each pt there are
2 normal unit vectors.

A natural choice of a unit normal vector:

$$\bar{n} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}$$

A natural choice of a unit normal vector

$$\bar{n} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}$$

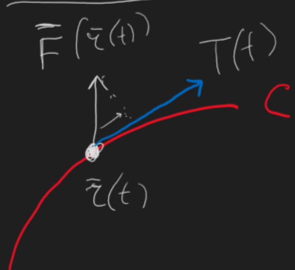
(The 2nd unit normal vector would be $-\bar{n}$).

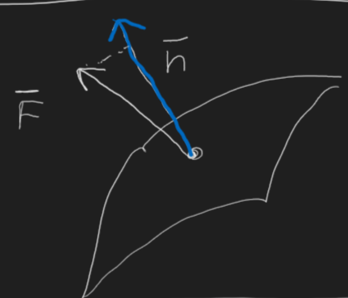
In particular case when S is graph of a f-n g: $\bar{n} = \dfrac{\langle -g_x, -g_y, 1\rangle}{|\langle -g_x, -g_y, 1\rangle|}$

upward orientation

downward orientation.

Curves	Surfaces

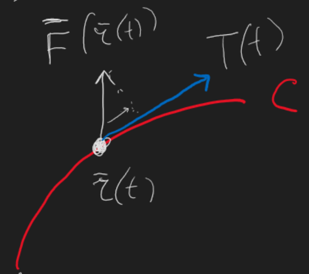


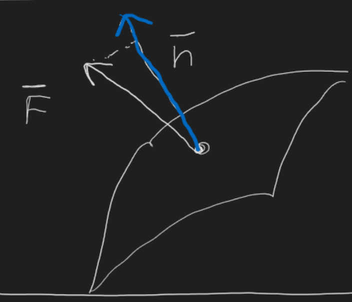


Def.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \bar{F}\cdot\bar{T}\, ds \qquad \boxed{\iint_S \bar{F}\cdot d\bar{S} = \iint_S \bar{F}\cdot\bar{n}\, dS}$$

Curves	Surfaces





Def.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \left(\bar{F}\cdot\bar{T}\right) ds$$

$$\boxed{\iint_S \bar{F}\cdot d\bar{S} = \iint_S \bar{F}\cdot\bar{n}\, dS}$$

Motivation for taking $\cdot\,\bar{n}$:

in computing how much liquid goes through S, only normal component of velocity vectors matter.

Def. The <u>surface integral</u> <u>of $\bar{F}$</u> <u>over S</u> is

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S (\bar{F}\cdot\bar{n})\, dS$$

It is also called the <u>flux</u> of F <u>across</u> S.

Def. The surface integral of F over S is

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S (\bar{F}\cdot\bar{n})\, dS.$$

It is also called the flux of F across S.

How to compute:

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S (\bar{F}\cdot\bar{n})\, dS$$

$$= \iint_D \left(\bar{F}\cdot\frac{\bar{r}_u\times\bar{r}_v}{\cancel{|\bar{r}_u\times\bar{r}_v|}}\right)\cancel{|\bar{r}_u\times\bar{r}_v|}\, dA$$

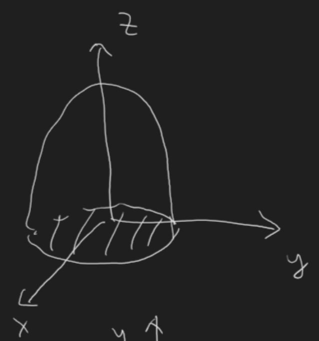
$$\boxed{\iint_S \bar{F}\cdot d\bar{S} = \iint_D \bar{F}\cdot(\bar{r}_u\times\bar{r}_v)\, dA}$$

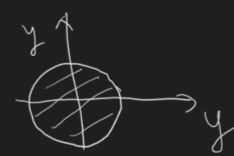
Case when S is a graph of g:

$\bar{F} = \langle P, Q, R\rangle$, $\bar{r}_u\times\bar{r}_v = \langle -g_x, -g_y, 1\rangle$

$$\boxed{\iint_S \bar{F}\cdot d\bar{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA}$$

Ex. Compute the flux of the vector field $\bar{F}(x,y,z) = y\bar{i} - x\bar{j} - z\bar{k}$ across the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$ oriented upward.





Solution

$$\bar{F}(x,y,z) = \langle y, -x, -z \rangle$$

$$= \langle y, -x, -1 + x^2 + y^2 \rangle.$$

$$\iint_S \bar{F} \cdot d\bar{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

$$= \iint_D \left(-y \cdot (-2x) - (-x) \cdot (-2y) - 1 + x^2 + y^2 \right) dA$$

$$= \iint_D \left(-1 + x^2 + y^2 \right) dA$$ = polar coordinates ... easy. □

Remark $\displaystyle\iint_{S \text{ oriented downward}} \bar{F} \cdot d\bar{S} = - \iint_{S \text{ oriented upward}} \bar{F} \cdot d\bar{S}$

Remark

$$\iint_S \bar{F}\cdot d\bar{S} = -\iint_S \bar{F}\cdot d\bar{S}$$

oriented downward — oriented upward

$$\bar{F} = \langle P, Q\rangle$$

Green's Th.:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

C: closed curve in $\mathbb{R}^2$

D: region bounded by C

Green's Th.:

$$\int_C \bar{F}\cdot d\bar{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

C: closed curve in $\mathbb{R}^2$

D: region bounded by C

Generalizing Green's Th. to $\mathbb{R}^3$

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_E \,?\; dV$$

S: closed ...

E: solid ...

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \,?\; dS$$

C: closed curve in $\mathbb{R}^3$

S: surface bounded by C

Generalizing Green's Th. to $\mathbb{R}^3$

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_E \, ? \, dV$$

S closed surface in $\mathbb{R}^3$

E solid bounded by S.

Divergence Th.
(Gauss-Ostrogradsky)

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \, ? \, dS$$

C closed curve in $\mathbb{R}^3$

S surface bounded by C

Stokes Th.

Divergence Theorem.

Def. Suppose $\bar{F} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$. The divergence of $\bar{F}$ is a f-n of 3 variables defined by

$$\operatorname{div} \bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Ex. Compute the divergence of

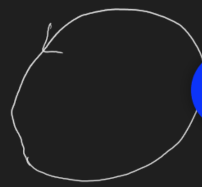
$$\bar{F}(x,y,z) = \underbrace{(\cos x + e^{y})}_{P}\,\bar{i} + \underbrace{e^{xz}}_{Q}\,\bar{j} - \underbrace{e^{z}\cos y}_{R}\,\bar{k}$$

Solution

$$\operatorname{div} \bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= -\sin x - e^{z}\cos y. \quad \square$$

Green's th.:

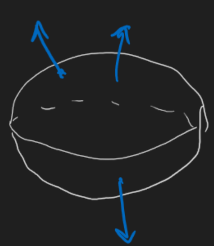


positively-oriented curve.

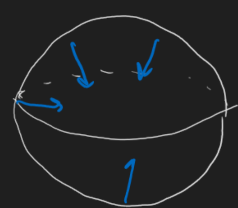
Divergence th.: closed positively-oriented surfaces.

S is <u>closed</u> if it is a boundary of a solid.

S is <u>positively oriented</u> if at each pt of S we choose the unit normal vector pointing outside.



positive orientation



negative orientation.

<u>Divergence Th. (Gauss-Ostrogradsky)</u> Suppose E is a solid bounded by a surface S given with the positive orientation. Suppose $\bar{F}$ is a vector field in $\mathbb{R}^3$. Then

$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_E \underline{\operatorname{div} F}\, dV.$$

Ex. Find the flux of the vector field

$\bar F(x,y,z) = \underbrace{z}_{P}\,\bar i + \underbrace{y}_{Q}\,\bar j + \underbrace{x}_{R}\,\bar k$ across the unit sphere.

Solution. Usual way:

$$\iint_S \bar F \cdot d\bar S = \iint_D \bar F \cdot (\bar r_\varphi \times \bar r_\theta)\, dA.$$

– long way.

Instead use Divergence Th.

$$\operatorname{div} \bar F = 0 + 1 + 0 = 1$$

$$\iint_S \bar F \cdot d\bar S = \iiint_{\text{unit ball}} 1 \, dV = \text{volume of unit ball} = \frac{4\pi}{3}. \quad \square$$

Remark. Unless stated otherwise, a closed surface is assumed to have positive orientation.

Ex. Compute $\iint_S \bar F \cdot d\bar S$, where

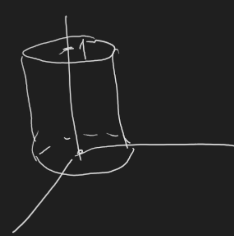
$$\bar F(x,y,z) = y z^3\, \bar i - e^{xz} \cos z\, \bar j + e^{xy} \sin x\, \bar k$$

a closed surface is assumed to have positive orienta… o

Ex. Compute $\iint_S \bar{F} \cdot d\bar{S}$, where

$$\bar{F}(x,y,z) = \underbrace{yz^3}_{P}\,\bar{i} - \underbrace{e^{xz}\cos z}_{Q}\,\bar{j} + \underbrace{e^{xy}\sin x}_{R}\,\bar{k}$$

and S is the surface of the region bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$.



Solution

$$\operatorname{div} \bar{F} = 0 + 0 + 0 = 0.$$

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_E \operatorname{div} \bar{F}\, dV = 0. \qquad \square$$