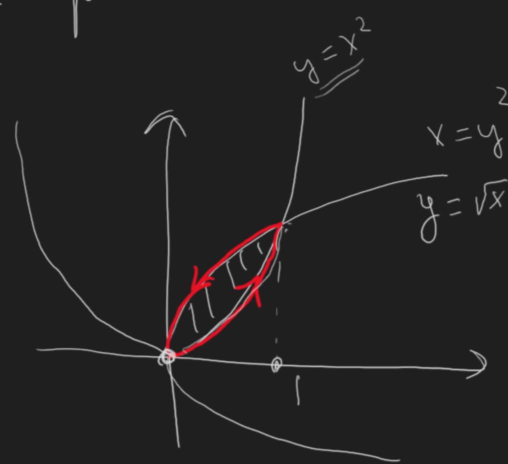
16.4 : 7, 13, 21.

7. Use Green's Th. to evaluate line integral

$$\int_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos y^2\right) dy,$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$, and C is positively oriented.



Solution

$$\int_C \underbrace{\left(y + e^{\sqrt{x}}\right)}_{P} dx + \underbrace{\left(2x + \cos y^2\right)}_{Q} dy$$

$$\underset{\text{Green}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (2-1)\, dA$$

$$= \iint_D dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = \int_0^1 \left[y\right]_{y=x^2}^{y=\sqrt{x}} dx$$

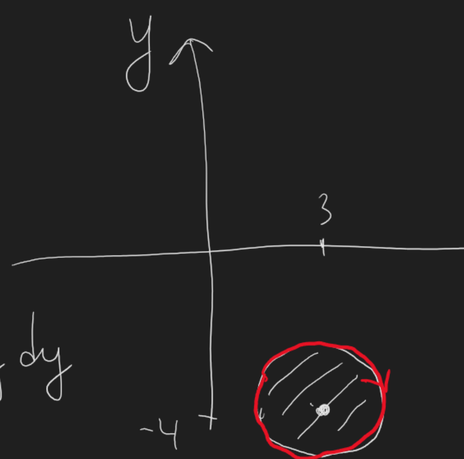
$$= \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left[\frac{2}{3} x^{3/2} - \frac{x^3}{3}\right]_{x=0}^{x=1}$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}. \quad \square$$

13. Evaluate $\int_C \bar{F} \cdot d\bar{r}$, where

$\bar{F}(x,y) = \langle y - \cos y,\ x \sin y \rangle$, C is t

circle $(x-3)^2 + (y+4)^2 = \underset{R^2}{4}$ oriented clockwis



Solution

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \overbrace{(y - \cos y)}^{P}dx + \overbrace{x \sin y}^{Q}\, dy$$

because C has negative orientation

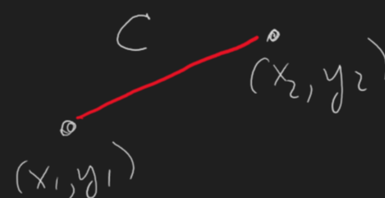
$$\overset{\text{Green}}{=} -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = -\iint_D \left(\cancel{\sin y} - (1 + \cancel{\sin y})\right) dA$$

$$= \iint_D dA = \text{area of } D = \pi \cdot 2^2 = 4\pi \qquad \square$$

21. (a) If C is the line segment connecting the pt (x_1, y_1) to the pt (x_2, y_2), show that $\int x\,dy - y\,dx = x_1 y_2 - x_2 y_1$

21. (a) If C is the line segment connecting the pt (x_1, y_1) to the pt (x_2, y_2), show that $\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1$.

Solution of (a):



Parametrization of C:

$$\vec{r}(t) = (1-t)\langle x_1, y_1\rangle + t\langle x_2, y_2\rangle = \langle (1-t)x_1 + t x_2,\ (1-t)y_1 + t y_2\rangle$$

$$= \langle (x_2 - x_1)t + x_1,\ (y_2 - y_1)t + y_1\rangle \qquad 0 \le t \le 1.$$

$$\int_C x\,dy - y\,dx = \int_0^1 ((x_2 - x_1)t + x_1)\, y'(t)\,dt - ((y_2 - y_1)t + y_1)\,x'(t)\,dt$$

$$= \int_0^1 \Big(((x_2 - x_1)t + x_1)(y_2 - y_1) - ((y_2 - y_1)t + y_1)(x_2 - x_1) \Big)\,dt$$

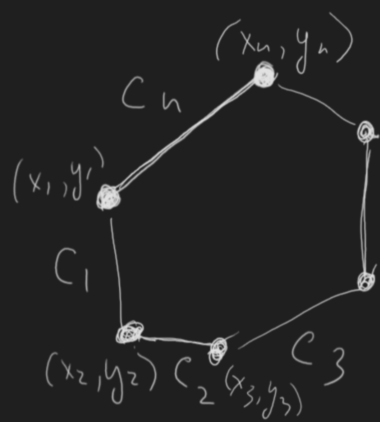
$$= \int_0^1 (x_1 y_2 - y_1 x_2)\,dt = x_1 y_2 - y_1 x_2. \qquad \square$$

(b) If the vertices of a polygon, in

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \frac{1}{2}\left[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \ldots + (x_n y_1 - x_1 y_n)\right].$$

Solution.



Area of region D bounded by C $= \frac{1}{2}\int_C x\,dy - y\,dx$

$$= \frac{1}{2}\left[\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \ldots + \int_{C_n} x\,dy - y\,dx\right]$$

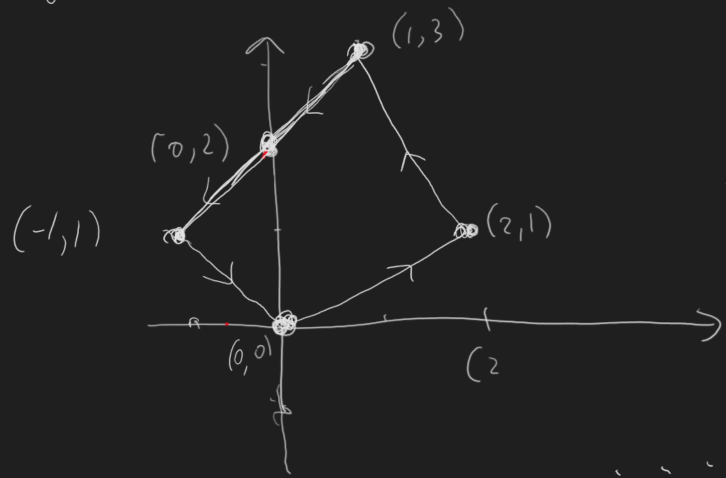
$\underline{\underline{(a)}}$ $\frac{1}{2}\left[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \ldots + (x_n y_1 - x_1 y_n)\right]$

$\square$

(c) Find the area of the pentagon

(c) Find the area of the pentagon with vertices $\underset{x_1\ \ y_1}{(0,0)}$, $\underset{x_2\ y_2}{(2,1)}$, $\underset{x_3\ y_3}{(1,3)}$, $(0,2)$ and $(-1,1)$

Solution



16.6: 33, 40

33 Find an equation of the tangent plane to the surface given by parametric equations $x = u+v$, $y = 3u^2$, $z = u-v$, at the pt $(2,3,0)$.

Solution First, we find the values of u, v corresponding to our pt $(2,3,0)$

$$2 = u+v, \quad 3 = 3u^2, \quad 0 = u-v$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$u = \pm 1 \qquad\qquad u = v$$

Solution: First, we find the values of u, v corresponding to our pt $(2,3,0)$:

$$2 = u+v, \quad 3 = 3u^2, \quad 0 = u-v$$

$$3 = 3u^2 \Downarrow u = \pm 1, \qquad 0 = u - v \Downarrow u = v$$

$$\Downarrow \quad u=1,\ v=1$$

A normal vector to the tangent plane at $(2,3,0)$ is

$$\bar{r}_u \times \bar{r}_v \ (1,1).$$

$$\bar{r}(u,v) = \langle u+v,\ 3u^2,\ u-v \rangle$$

$$\bar{r}_u(u,v) = \langle 1,\ 6u,\ 1 \rangle \qquad \bar{r}_u(1,1) = \langle 1, 6, 1 \rangle$$

$$\bar{r}_v(u,v) = \langle 1,\ 0,\ -1 \rangle. \qquad \bar{r}_v(1,1) = \langle 1, 0, -1 \rangle$$

$$\bar{r}_u(1,1) \times \bar{r}_v(1,1) = \det \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 6 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= -6\bar{i} + 2\bar{j} - 6\bar{k}.$$ — normal vector to the tangent plane at $(2,3,0)$.

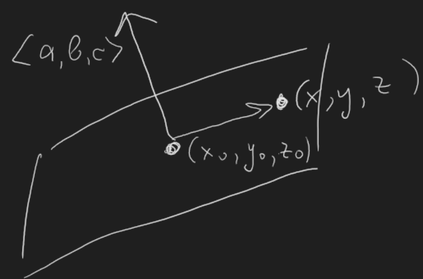
$$= \langle -6, 2, -6 \rangle.$$

Equation of a plane if we know a pt (x_0, y_0, z_0) on the plane and plane at $(2,3,0)$.

$= \langle -6, 2, -6 \rangle.$

Equation of a plane if we know a pt (x_0, y_0, z_0) on the plane and a normal vector $\langle a, b, c \rangle$:

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$



In our case

$$-6(x-2) + 2(y-3) - 6z = 0.$$

$$-6x + 2y - 6z + 6 = 0.$$

$$3x - y + 3z - 1 = 0.$$

□.

40. Find the area of the part of the plane with vector equation

$$\vec{r}(u,v) = \langle u+v, 2-3u, 1+u-v \rangle,$$

$$0 \le u \le 2, \quad -1 \le v \le 1.$$

40. Find the area of the part of the plane with vector equation

$$\bar{r}(u,v) = \langle u+v,\ 2-3u,\ 1+u-v \rangle,$$

$$0 \le u \le 2,\quad -1 \le v \le 1.$$

Solution

$$\text{Area of a surface } S = \iint_S 1\, dS$$

$$= \iint_D 1 \cdot |\bar{r}_u \times \bar{r}_v|\, dA$$

$$\bar{r}_u = \langle 1, -3, 1 \rangle$$

$$\bar{r}_v = \langle 1, 0, -1 \rangle$$

$$\bar{r}_u \times \bar{r}_v = \det \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

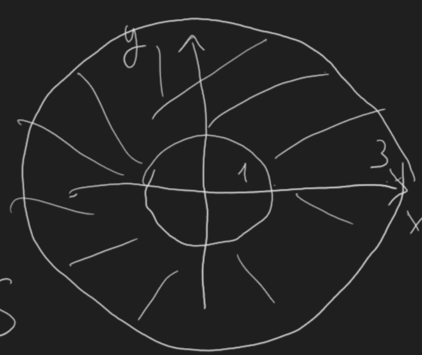
$$= 3\bar{i} + 2\bar{j} - 3\bar{k}$$

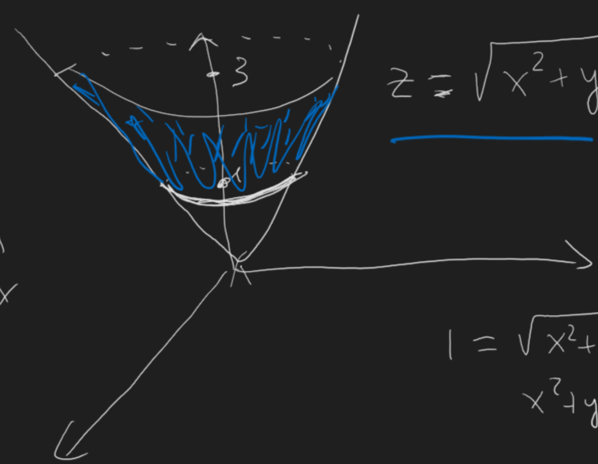
$$|\bar{r}_u \times \bar{r}_v| = \sqrt{9+4+9} = \sqrt{22}.$$

$$\text{area} = \iint_D \sqrt{22}\, dA = \int_{-1}^{1}\int_{0}^{2} \sqrt{22}\, du\, dv = \sqrt{22}\cdot 4.$$

16.7 : 11 Compute $\iint_S x^2 z^2 \, dS$, where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Solution.





$z = \sqrt{x^2+y^2}$

$1 = \sqrt{x^2+y^2}$
$x^2+y^2 = 1$

$3 = \sqrt{x^2+y^2}$
$x^2+y^2 = 9$

$$\iint_S x^2 z^2 \, dS$$

$$= \iint_D x^2 (x^2+y^2) \sqrt{1 + z_x^2 + z_y^2} \, dA$$

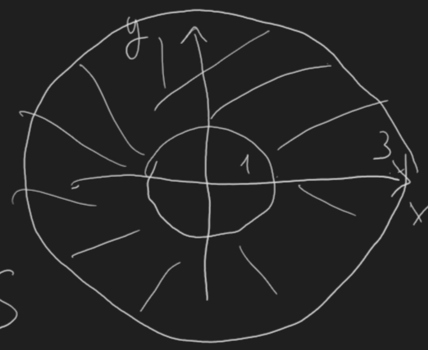
$$= \iint_D x^2 (x^2+y^2) \sqrt{1 + \frac{x^2 + y^2}{x^2+y^2}} \, dA$$

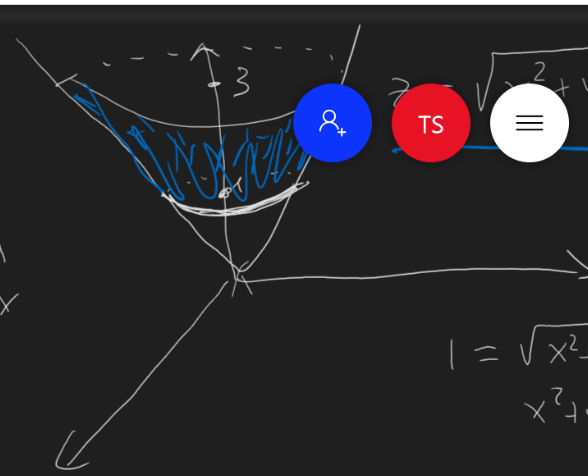
$$= \iint_D x^2 (x^2+y^2) \sqrt{2} \, dA = \int_0^{2\pi}\int_1^3 r^2 \cos^2\theta \cdot r^2 \sqrt{2} \, r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi}\int_1^3 r^5 \cos^2\theta \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{r^6 \cos^2\theta}{6} \right]_{r=1}^{r=3} d\theta$$

2π

Solution.





$z = \sqrt{x^2+y^2}$

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$x^2+y^2 = 9$

$$\iint_S x^2 z^2 \, dS$$

$$= \iint_D x^2 (x^2+y^2) \sqrt{1 + z_x^2 + z_y^2} \, dA$$

$$= \iint_D x^2 (x^2+y^2) \sqrt{1 + \frac{x^2 + y^2}{x^2+y^2}} \, dA$$

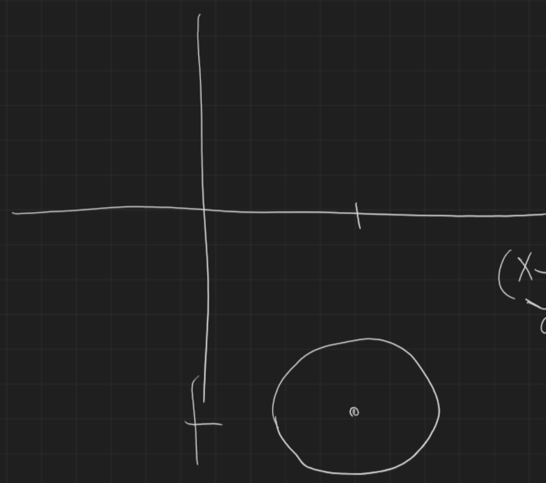
$$= \iint_D x^2 (x^2+y^2) \sqrt{2} \, dA = \int_0^{2\pi}\int_1^3 r^2 \cos^2\theta \cdot r^2 \sqrt{2} \, r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi}\int_1^3 r^5 \cos^2\theta \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{r^6 \cos^2\theta}{6} \right]_{r=1}^{r=3} d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \cos^2\theta \left(\frac{3^6 - 1}{6} \right) d\theta = \frac{\sqrt{2}\,(3^6-1)}{6} \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$= \frac{\sqrt{2}\,(3^6-1)}{6} \left[\frac{1}{2}\theta + \frac{\sin(2\theta)}{4} \right]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{\sqrt{2}\,(3^6-1)}{6} \pi \quad \square$$



$$(\underbrace{x-3}_{a})^2 + (\underbrace{y-4}_{b})^2 = 4$$

$$a^2 + b^2 = 4.$$

$$a = 2\cos t, \quad b = 2\sin t$$

$$0 \le t \le 2\pi.$$

$$x = a + 3 = 2\cos t + 3$$

$$y = b + 4 = 2\sin t + 4$$

$$0 \le t \le 2\pi.$$