16.7.21 Evaluate $\iint_S F \cdot dS$ with $F(x,y,z) = z e^{xy} \vec{i} - 3z e^{xy} \vec{j} + xy \vec{k}$

S : $x = u+v$, $y = u-v$, $z = 1+2u+v$ with $0 \leq u \leq 2$, $0 \leq v \leq 1$

Recall S: $r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ 16.6.6

$$\left.\begin{aligned} r_u(u,v) &= \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \\ r_v(u,v) &= \ldots \end{aligned}\right\} \text{tangent vectors to } S$$

$r_u \times r_v$ is a normal vector to S

16.7.8 $\iint_S \vec{F} \cdot \vec{dS} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \overrightarrow{(r_u \times r_v)}\, dA$

$\hookrightarrow$ domain of the parametrisation

$$r_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$

$$= \frac{\partial(u+v)}{\partial u}\vec{i} + \frac{\partial(u-v)}{\partial u}\vec{j} + \frac{\partial(1+2u+v)}{\partial u}\vec{k}$$

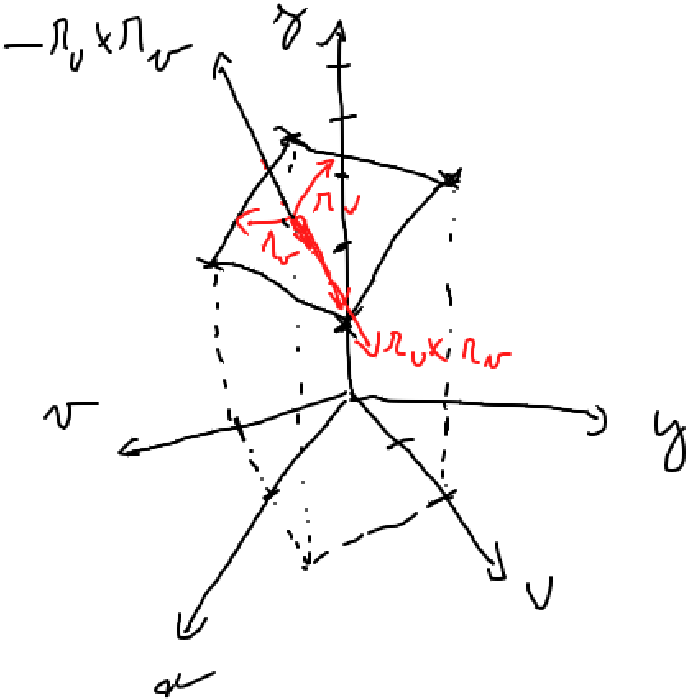
$$= \vec{i} + \vec{j} + 2\vec{k}$$

$$r_v = \frac{\partial}{\partial v} \cdots = \vec{i} - \vec{j} + \vec{k}$$

$$r_u \times r_v = \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \end{array}$$

$$r_u \times r_v = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = (1\times1-(-1)\times2)\vec{i} + (2\times1 - 1\times1)\vec{j} + (1\times(-1) - 1\times1)\vec{k}$$

$$= 3\vec{i} + \vec{j} - 2\vec{k}$$



$-r_u \times r_v$
z
r_v
r_u
$r_u \times r_v$
v
y
u
x

orientation

$$\iint_S F\cdot dS = \int_D F\cdot(-r_u \times r_v)\, dA = -\int_0^2\int_0^1 3\times \cancel{z(u,v)\, e^{x(u,v)\,y(u,v)}} + 1\times\cancel{(-3z(u,v))e^{xy}} - 2xy \; du\, dv$$

⚠ x, y, z are functions of (u,v)

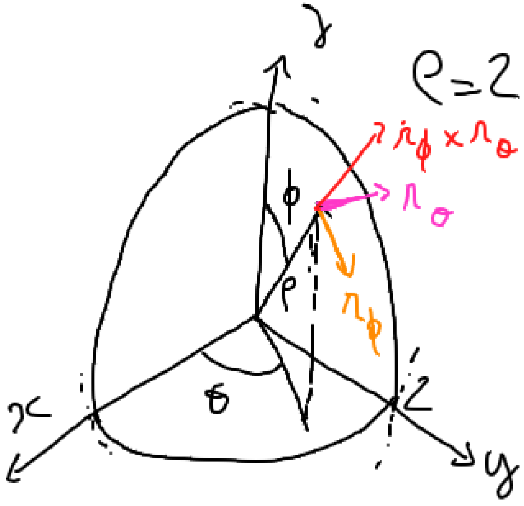
$$\iint_S F\cdot dS = +\int_0^2\int_0^1 2\,(u+v)(u-v)\, du\, dv = +2\left(\int_0^2 u^2 du \int_0^1 dv - \int_0^2 du \int_0^1 v^2 dv\right)$$

$$= +2\left(\frac{2^3}{3}\times 1 - 2\times\frac{1^3}{3}\right) = +4$$

16.7.29

$F(x,y,z) = x\vec{i} - z\vec{j} + y\vec{k}$

S: first-octant of sphere of radius 2 with orientation toward the origin



Recall 15.8.1 parametrisation of the sphere

$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho\cos\phi$

$r(\phi,\theta) = x(\phi,\theta)\vec{i} + y(\phi,\theta)\vec{j} + z(\phi,\theta)\vec{k}$ (16.6.4)

$r_\phi = \dfrac{\partial r}{\partial \phi} \qquad r_\theta = \dfrac{\partial r}{\partial \theta}$

$r_\phi \times r_\theta = \rho^2(\sin^2\phi\cos\theta\,\vec{i} + \sin^2\phi\sin\theta\,\vec{j} + \sin\phi\cos\phi\,\vec{k})$ (16.6.10)

orientation

$\iint_S F\cdot dS = \iint_D F\cdot(r_\phi \times r_\theta)\,dA = -\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \rho\sin\phi\cos\theta \times \rho^2\sin^2\phi\cos\theta - \rho\cos\phi\,\cancel{\rho^2\sin^2}\phi\sin\theta + \rho\sin\phi\sin\theta\,\cancel{\rho^2\sin}\phi\cos\phi\,d\phi\,d\theta$

$$\iint F\cdot dS = -\rho^3 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^3\phi \, \cos^2\theta \; d\theta \, d\phi$$

derive $\to$ $-\sin\phi - \frac{3}{3}(-\sin\phi)\cos^2\phi = \sin\phi\,(\cos^2\phi - 1)$

$= \sin\theta\,(-\sin^2\phi)$

$= -\sin^3\phi$

$$= -\rho^3 \int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \rho^3 \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} \, d\theta$$

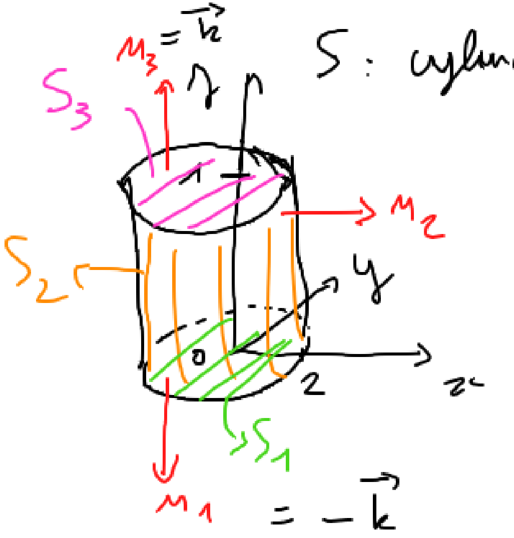
$$= -\rho^3 \left(1 - \frac{1}{3}\right)\left(\frac{\pi}{4} + \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}\right)$$

$$= -8 \times \frac{2}{3} \times \frac{\pi}{4}$$

$$= -\frac{4\pi}{3}$$

16.7.43 $\rho \vec{v} = \vec{F} = \rho(z\vec{i} + y^2\vec{j} + x^2\vec{k})$

S : cylinder $x^2+y^2=4$, $0 \le z \le 1$



$$\iint_S F\cdot dS = \iint_{S_1} F\cdot dS + \iint_{S_2} F\cdot dS + \iint_{S_3} F\cdot dS$$

$$= \iint_{S_1} F\cdot n_1 \, dS + \iint_{S_3} F\cdot n_3 \, dS + \iint_{S_2} F\cdot dS$$

$$= \cancel{\iint_{S_1} x^2 \times (-1)\, dS} + \cancel{\iint_{S_3} x^2 \, dS} + \iint_{S_2} F\cdot dS$$

since it doesn't depend on z

<u>recall</u> $x = 2\cos\theta \ , \ y = 2\sin\theta \ , \ z = z$ (16.6.6)

$r_\theta = -2\sin\theta \, \vec{i} + 2\cos\theta \, \vec{j}$

$r_z = \vec{k}$

$$r_\theta \times r_z = \begin{vmatrix} -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos\theta \, \vec{i} + 2\sin\theta \, \vec{j}$$

$$\iint_{S_2} F \cdot dS = \int_0^{2\pi}\int_0^1 z \times 2\cos\theta + (2\sin\theta)^2 \, 2\sin\theta \, dz \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \times 2\cos\theta + 1 \times 8\sin^3\theta \, d\theta$$

$$= \left[\sin\theta\right]_0^{2\pi} + 8\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{2\pi}$$

$$= 0 + 8 \times 0 = 0$$

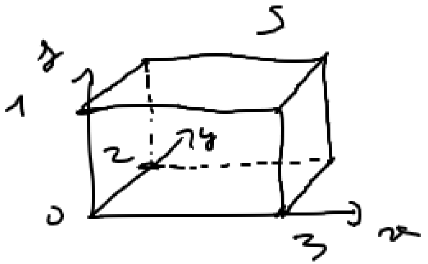
$$\iint_S F \cdot dS = \iint_S \rho \, \vec{v} \cdot \vec{dS} = 0$$

16.9.5

Recall divergence theorem $\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$

$$F(x,y,z) = xy e^{z} \vec{i} + xy^2 z^3 \vec{j} - y e^{z} \vec{k}$$

S: box of coordinate planes and $x = 3$, $y = 2$, $z = 1$



$F = P\vec{i} + Q\vec{j} + R\vec{k}$ then $\operatorname{div} F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV = \int_0^3 \int_0^2 \int_0^1 \cancel{ye^{z}} + 2y\,xz^3 \quad - \cancel{ye^{z}} \; dz\,dy\,dx$$

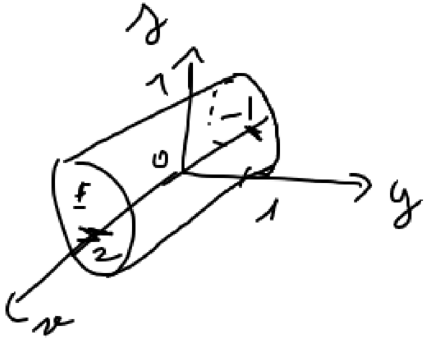
$$\iint_S F \cdot dS = 2\int_0^3 \int_0^2 \int_0^1 x\,y\,z^3\,dz\,dy\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^3 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{z^4}{4}\right]_0^1$$

$$= 2\,\frac{9}{2}\,\frac{4}{2}\,\frac{1}{4} = \frac{9}{2}$$

16.9.7 $F(x,y,z) = 3xy^2\,\vec{i} + xe^z\,\vec{j} + z^3\,\vec{k}$

S: cylinder $y^2+z^2=1$ and planes $x=-1$, $x=2$



E is a simple solid region, S the boundary surface of E given with positive (outward) orientation.
F is a vector field whose component functions have continuous partial derivatives on an open region that contains E

So we can apply divergence theorem

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

$$\iint_S F \cdot dS = \iiint_E 3y^2 + 0 + 3z^2 \, dV$$

$$= 3\int_{-1}^{2} \iint_{D_x} y^2 + z^2 \, dy\, dz \, dx$$

$$= 3\int_{-1}^{2} \int_0^1 \int_0^{2\pi} r^2 \, r \, d\theta \, dr \, dx$$

$$= 3\int_{-1}^{2} 2\pi \left[\frac{r^4}{4}\right]_0^1 dx$$

$$= \frac{6\pi}{4} \times 3 = \frac{9\pi}{2}$$

$$\begin{cases} y = r\cos\theta \\ z = r\sin\theta \end{cases}$$

$$y^2 + z^2 = r^2$$

16.5.23 Verify that $\operatorname{div} \vec{E} = 0$ for $\vec{E}(\vec{r}) = \varepsilon Q \dfrac{\vec{r}}{|\vec{r}|^3}$ $\quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

- $f(g(u)) = g'(u)\, f'(g(u))$

- $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - v'u}{v^2}$

$$= \varepsilon Q \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial(E\cdot\vec{i})}{\partial x} = \varepsilon Q \frac{\left[(x^2+y^2+z^2)^{3/2} - \frac{3}{2}\, 2x\,(x^2+y^2+z^2)^{1/2} \times x\right]}{(x^2+y^2+z^2)^3}$$

$$\operatorname{div}(E) = \frac{\varepsilon Q}{(x^2+y^2+z^2)^3}\left(3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)\right)$$

$$= \frac{\varepsilon Q}{(x^2+y^2+z^2)^3}\left(3(x^2\cancel{+y^2}+z^2)^{3/2} - 3(x^2\cancel{+y^2}+z^2)^{3/2}\right) = 0$$