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Previous lecture:

$$\iint_S \bar{F}\cdot d\bar{S} \overset{def}{=} \iint_S \bar{F}\cdot\bar{n}\, dS =$$

flux of $\bar{F}$ across S

formula for computation

$$= \iint_D \bar{F}(\bar{r}(u,v))\cdot(\bar{r}_u \times \bar{r}_v)\, dA$$

Phys. sense: rate of flow of some fluid (or heat...) across the surface.

Divergence Th.

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_E \operatorname{div}\bar{F}\, dV$$

S: closed positively oriented surface

E: solid bounded by S

$\bar{F} = \langle P, Q, R\rangle$

$$\operatorname{div}\bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

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$$\operatorname{div} \bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Green's Th.: $\displaystyle\int_C \bar{F} \cdot d\bar{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

C: closed, positively oriented

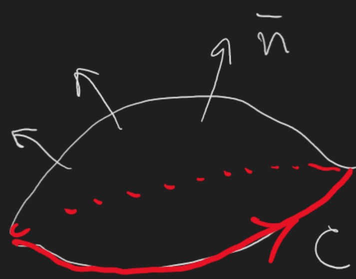
D: region bounded by C

Stokes' Theorem.

Def. If $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ is a vector field, then the curl of $\bar{F}$ is the vector field defined by

$$\operatorname{curl} \bar{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\bar{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\bar{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\bar{k}.$$

Why called the curl: if $\bar{F}$ is a velocity field of some fluid, then curl $\bar{F}$ contains info about rotations (whirlpools). Namely, the axis about which whirlpool rotates [illegible]

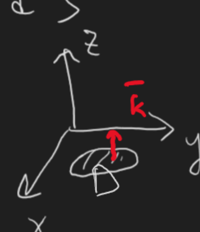


Suppose C is the boundary curve of S. We choose orientation of C such that it agrees with orientation of S in the following sense: when we walk along C in the positive direction with head pointing in the direction of $\bar n$, S should be on our left.

<u>Stokes' Theorem</u> Suppose C is the boundary curve of S and orientations of C and S agree. Then

$$\int_C \bar F \cdot d\bar r = \iint_S \text{curl } \bar F \cdot d\bar S$$

Suppose C is a plane curve and S is a flat region.

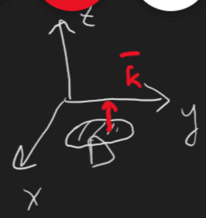


$\bar F$ is a v-f field in $\mathbb{R}^2$.

$$\bar F(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$$

left-hand side of Stokes: $\int P\,dx + Q\,dy$

Suppose C is a plane curve
is a flat region.



$\bar{F}$ is a v-f field in $\mathbb{R}^3$.

$$\bar{F}(x,y,z) = \langle P(x,y),\ Q(x,y),\ 0 \rangle$$

Left-hand side of Stokes: $\int_C P\,dx + Q\,dy$

Right-hand side of Stokes: $\iint_D \langle 0, 0, \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}\rangle \cdot \langle 0,0,1 \rangle\, dA$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Therefore Green's Th. is a particular case of Stokes' Theorem.

Review of the course.

- Limits and continuity – definitions are the same as for f-s of 1 var., but more methods for showing $\lim_{(x,y)\to(a,b)} \nexists$:

Review of the course.

- Limits and continuity – definitions are the same as for f-s of 1 var., but more methods for showing $\lim_{(x,y)\to(a,b)} \nexists$:

  approaching (x,y) along different curves should give the same limit, otherwise $\lim_{(x,y)\to(a,b)} \nexists$.

- Differention.
  - Partial derivatives. Higher partial derivatives. $f_{xy} = f_{yx}$.
  - Differentiability (geometrically: tangent plane $\exists$ at each pt).
  - Chain Rule. $f(x,y)$, $x = x(u,v)$, $y = y(u,v)$.

    $$\frac{\partial f}{\partial u} = \dots, \quad \frac{\partial f}{\partial v} = \dots$$

  - Gradient $\nabla f = \langle f_x, f_y \rangle$

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    $$\frac{\partial f}{\partial u} = \cdots, \quad \frac{\partial f}{\partial v} = \cdots$$

  - Gradient $\nabla f = \langle f_x, f_y \rangle$
  - Directional derivatives

    $$D_{\bar{u}} f = \nabla f \cdot \bar{u}$$

    unit vector

- Max/min
  - local max/min occur in critical pts (that is pts at which $\boxed{\nabla f = 0}$ or $\nabla f \nexists$.)

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- Max / min

  - local max/min occur in critical pts (that is pts at which $\boxed{\nabla f = 0}$ or $\nabla f \nexists$.)

    2nd Derivative Test.

  - absolute (global) max/min.

    Extreme Value Th.: they $\exists$ on a closed bounded set.

    How to find?

    They can be either at critical pts or on the boundary.

    How to find abs. max/min on the boundary? Either it is easy or we use

  - Lagrange Multiplier Method.

- Integration.

• Integration.

• Double integrals of f-s of 2 var. over regions in $\mathbb{R}^2$.

$$\iint_D f(x,y)\,dA = \int_{\dots}^{\dots}\int_{\dots}^{\dots} f(x,y)\,dx\,dy$$

$dy\,dx$

order depends on D.

area of $D = \iint_D dA$.

• change of variables in double int-s (Jacobian).

• polar coordinates

$$\iint_D f(x,y)\,dA = \iint_D \dots\, r\,dr\,d\theta$$

$d\theta\,dr$.

• Triple integrals of f-s of 3 var. over solids in $\mathbb{R}^3$.

$$\iiint_E f(x,y,z)\,dV = \int_{\dots}^{\dots}\int_{\dots}^{\dots}\int_{\dots}^{\dots} f(x,y,z)\,dx\,dy\,dz$$

$dz\,dx\,dy$

$\dots$

volume of $E = \iiint_E dV$.

$$\text{volume of } E = \iiint_E dV.$$

$$\text{mass of } E = \iiint_E \rho(x,y,z)\,dV$$

density f-n.

(x_0, y_0, z_0) – center of mass of E.

$$x_0 = \frac{\iiint_E x\,\rho(x,y,z)\,dV}{\text{mass}}, \quad y_0 = \dots, \quad z_0 = \dots$$

In particular case when E lies under the graph of f above region D in (x,y)-plane,

$$\text{volume of } E = \iint_D f(x,y)\,dA.$$

- Change of variables (Jacobian).

  - Spherical coordinates

$$\iiint_E f(x,y,z)\,dV = \iiint_E \dots \boxed{\rho^2 \sin\varphi}\, d\rho\, d\varphi\, d\theta$$

order depends on E.

• Line integrals (integrals over curves).

Can integrate a f-n of 2 var. over a plane curve and a f-n of 3 var. over a space curve.

Suppose C is given by parametrization
$\bar{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$

$$\bullet \int_C f\,ds \overset{\text{f-la for computing}}{=\!=} \int_a^b f(\bar{r}(t))\,|\bar{r}'(t)|\,dt$$

mass of C where C is made of material of density f.

$$\text{length of } C = \int_C ds = \int_a^b |\bar{r}'(t)|\,dt.$$

$$\bullet \int_C f\,dx \overset{\text{f-la for computing}}{=\!=} \int_a^b f(\bar{r}(t))\,x'(t)\,dt$$

$$\bullet \int_C f\,dy = \int_a^b f(\bar{r}(t))\,y'(t)\,dt$$

Therefore computing line integrals reduces to computing usual integrals of f-s of 1 var.

• Integrals of vector fields over curves

- Integrals of vector fields over curves $\bar{F} = \langle P, Q \rangle$ or $\langle P, Q, R \rangle$, then

work done by force fields $= \int_{C} \bar{F} \cdot d\bar{r} = \int_{C} P\,dx + Q\,dy \quad (+ R\,dz)$.

Conservative v-r fields, Fund. Th. for line int-s: $\int_{C} \nabla f \cdot d\bar{r} = f(r(b)) - f(r(a))$.

Independence on path.

Green's Th.: $\int_{\substack{C \\ \text{closed}}} \bar{F} \cdot d\bar{r} = \int_{C} P\,dx + Q\,dy$

$$= \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

area of $D = \int_{C} x\,dy = -\int_{C} y\,dx = \frac{1}{2} \int_{C} x\,dy - y\,dx$.

C: boundary curve of D

- Integrals over surfaces.

  - Integrals of f-s of 3 var. over surfaces.

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- Integrals over surfaces

- Integrals of f-s of 3 var. over surfaces.

$\bar{r}(u,v)$ – param-n of S.

$$\iint_S f(x,y,z)\, dS \overset{\text{f-la}}{=} \iint_D f(\bar{r}(u,v))\, |\bar{r}_u \times \bar{r}_v|\, dA$$

area of $S = \iint_S dS$

If S is a graph, we get simpler f-la:

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,z(x,y)) \cdot \sqrt{1+z_x^2+z_y^2}\, dA.$$

- Surface int-s of vector fields.

- Divergence Theorem.

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