Microsoft Whiteboard

Monday: 10-12 zoom

(in case one has questions).

also by e-mail.

Example exam.

① Determine the set of pts at which the f-n is contin.

$$f(x,y) = \begin{cases} \dfrac{x^2 y^5}{2x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0). \end{cases}$$

Solution At each $(x,y) \neq (0,0)$ f is continuous, because the ratio of 2 continuous f-s is continuous on its domain.

$$\frac{g(x,y)}{h(x,y)}$$

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Solution At each $(x, y) \neq (0,0)$ f is continuous, because the ratio of 2 continuous f-s is continuous on its domain.

$$\frac{g(x,y)}{h(x,y)}$$

Checking continuity at $(0,0)$:

$$\lim_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} f(0,0) = 1.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y^5}{2x^2+y^2}$$

Approaching $(0,0)$ by the curve $x=0$:

$$\frac{x^2 y^5}{2x^2+y^2} = \frac{0}{y^2} = 0 \neq 1$$

$\Longrightarrow$ f is not contin. at $(0,0)$ $\square$

We could find $\lim_{(x,y)\to(0,0)} \frac{x^2 y^5}{2x^2+y^2}$

$$\left| \frac{x^2}{2x^2+y^2} \cdot \underset{\downarrow \atop 0}{y^5} \right| \le \left| \frac{1}{2} y^5 \right| \to 0.$$

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$(x,y)\to(0,0)$ $\qquad$ $(x,y)\to(0,0)$ $\quad 2x^2+y^2$

Approaching $(0,0)$ by the curve $x=0$:

$$\frac{x^2y^5}{2x^2+y^2} = \frac{0}{y^2} = 0 \neq 1$$

$\Longrightarrow f$ is not contin. at $(0,0)$ □

We could find $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^5}{2x^2+y^2}$

$$\left|\frac{x^2}{2x^2+y^2}\cdot \underset{\substack{\downarrow\\0}}{y^5}\right| \le \left|\frac{1}{2}y^5\right| \to 0.$$

or alternatively

$$\underset{\substack{\downarrow\\0}}{-\frac{1}{2}y^5} \le \frac{x^2y^5}{2x^2+y^2} \le \underset{\substack{\downarrow\\0}}{\frac{1}{2}y^5}$$

$$\frac{x^2y^5}{2x^2+y^2} \xrightarrow{\text{Squeeze Th.}} 0.$$

$$\frac{x^2y^5}{2x^2+y^2} = \frac{r^2\cos^2\varphi\, r^5\sin^5\varphi}{2r^2\cos^2\varphi + r^2\sin^2\varphi}$$

$$= r^3\frac{\cos^2\varphi\sin^5\varphi}{2\cos^2\varphi+\sin^2\varphi} \to ?$$

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(2) Let $f(x,y) = x^2 - y^2 - 2x$

a) Find and classify critical pts

b) Find the absolute max and min values of f on the region

$$D = \{(x,y) \mid x^2 + y^2 \le 9\}$$

Solution a) $\nabla f = \langle 2x-2, -2y \rangle = 0$.

$$2x - 2 = 0, \quad -2y = 0$$

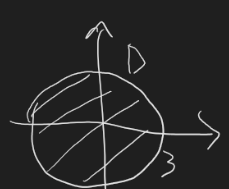
$$x = 1, \quad y = 0.$$

1 critical pt: $(1,0)$.

$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{xy} = 0$.

$$D(1,0) = \left(f_{xx} f_{yy} - (f_{xy})^2\right)(1,0) = -4 < 0$$

$\Longrightarrow$ $(1,0)$ is a saddle pt.

b) 

Pts of abs. max and min must be either at crit. pts inside D or on the boundary.

b) 

Pts of abs. max ... must be either at crit. pts inside D or on the boundary.

We find abs. max/min on the boundary.

So we need to find abs. max/min subject to the restriction $x^2+y^2=9$.

$g(x,y)=x^2+y^2-9$

Lagrange Multiplier Method:

$$\nabla f = \lambda \nabla g$$
$$g(x,y)=0$$

$$\langle 2x-2, -2y\rangle = \lambda \langle 2x, 2y\rangle$$
$$x^2+y^2-9=0$$

$$\Rightarrow \begin{cases} 2x-2=\lambda\cdot 2x \\ -2y=\lambda\cdot 2y \\ x^2+y^2=9 \end{cases}$$

$\Rightarrow y(2\lambda+2)=0 \Rightarrow$ either $y=0$ or $\lambda=-1$.

Case 1: $y=0$

$$2x-2=\lambda\cdot 2x \Rightarrow 2x(1-\lambda)=2$$
$$x^2=9 \Rightarrow x=\pm 3$$

$$1-\lambda=\frac{1}{x}$$
$$\lambda = 1-\frac{1}{x}$$
$$\lambda = 1\mp\frac{1}{3}$$

we don't need it (didn't have to find).

$(3,0)$ and $(-3,0)$

Case 2: $\lambda=-1$.

$$2x-2=-2x$$
$$4x=2$$

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$(3,0)$ and $(-3,0)$

$1-\lambda = \frac{1}{x}$

$\lambda = 1 - \frac{1}{x}$

$\lambda = 1 \mp \frac{1}{3}$

we don't need it
(didn't have to find).

Case 2: $\lambda = -1$. $\quad 2x - 2 = -2x$

$4x = 2$

$x = \frac{1}{2}$

$$\frac{1}{4} + y^2 = 9 \Rightarrow y^2 = \frac{35}{4} \Rightarrow y = \pm\frac{\sqrt{35}}{2}.$$

$$\left(\frac{1}{2}, \frac{\sqrt{35}}{2}\right) \text{ and } \left(\frac{1}{2}, -\frac{\sqrt{35}}{2}\right).$$

$f(1,0) = -1, \quad f\left(\frac{1}{2}, \frac{\sqrt{35}}{2}\right) = -9\frac{1}{2}$ abs. min value

$f\left(\frac{1}{2}, -\frac{\sqrt{35}}{2}\right) = -9\frac{1}{2}$

$f(3,0) = 3$

$f(-3,0) = 15$ - abs. max value.

(3) Compute the double integral

$\iint \frac{x+y}{...}dA$ where D is the square

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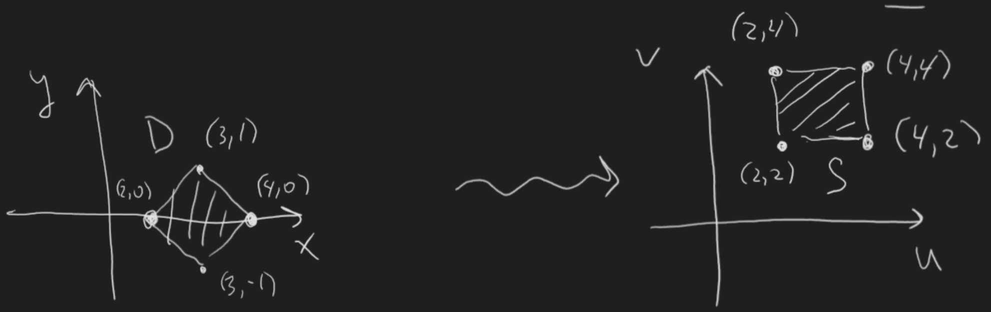
③ Compute the double integral

$$\iint_D \frac{x+y}{x-y}\,dA,$$ where D is the square with vertices $(2,0),(3,-1),(4,0),(3,1)$.

Solution

$$u := x+y,\quad v := x-y \implies x = \frac{u+v}{2},\quad y = \frac{u-v}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}.$$



$$\iint_D \frac{x+y}{x-y}\,dA = \iint_S \frac{u}{v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \frac{1}{2}\int_2^4\int_2^4 \frac{u}{v}\,du\,dv = \frac{1}{2}\int_2^4 \frac{1}{v}\left[\frac{u^2}{2}\right]_{u=2}^{u=4} dv$$

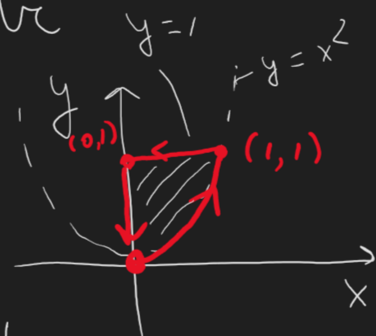
$$= \frac{6}{2}\int_2^4 \frac{1}{v}\,dv = 3\left[\ln v\right]_{v=2}^{v=4} = 3(\ln 4 - \ln 2)$$

□

(4) Compute the work done by the force field $\bar{F}(x,y) = x^2y^2\,\hat{i} + xy\,\hat{j}$ in moving a particle along C consisting of the arc of parabola $y = x^2$ going from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.

Solution work $= \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C \overbrace{x^2y^2}^{P}\,dx + \overbrace{xy}^{Q}\,dy$$



Green

$$\iint_D (y - 2x^2y)\,dA = \int_0^1\int_{x^2}^1 (y - 2x^2y)\,dy\,dx$$

$$= \int_0^1\int_{x^2}^1 (1-2x^2)\,y\,dy\,dx = \int_0^1 (1-2x^2)\left[\frac{y^2}{2}\right]_{y=x^2}^{y=1} dx$$

$$= \frac{1}{2}\int_0^1 (1-2x^2)(1-x^4)\,dx = \frac{1}{2}\int_0^1 (1-2x^2-x^4+2x^6)\,dx$$

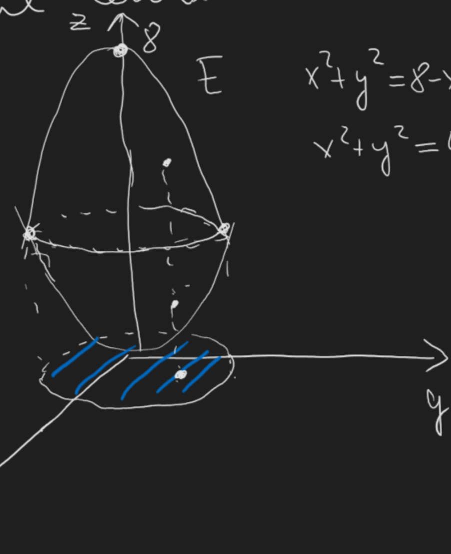
$$= \frac{1}{2}\left[x - \frac{2x^3}{3} - \frac{x^5}{5} + \frac{2x^7}{7}\right]_{x=0}^{x=1} =$$

$$= \frac{1}{2}\left(1 - \frac{2}{3} - \frac{1}{5} + \frac{2}{7}\right) = \quad \square$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} + \frac{2}{7}\right) = \frac{1}{2}\cdot\frac{35-21+30}{105} = \frac{22}{105}$$

5) Find the center of mass of the solid of density $\rho(x,y,z)=1$ enclosed by paraboloids $z=x^2+y^2$ and $z=8-x^2-y^2$

Solution (x_0,y_0,z_0) – the center of mass.



$x^2+y^2=8-x^2-y^2$

$x^2+y^2=4$

By symmetry arguments the center of mass lies on the z-axis, so $x_0=y_0=0$.

$$z_0=\frac{\iiint_E z\,\rho(x,y,z)\,dV}{\iiint_E \rho(x,y,z)\,dV}=\frac{\iiint_E z\,dV}{\iiint_E dV}$$

$$E=\{(x,y,z)\mid x^2+y^2\le 4,\ x^2+y^2\le z\le 8-x^2-y^2\}$$
$$=\{(r,\theta,z)\mid r\le 2,\ r^2\le z\le 8-r^2\}.$$

$$\iiint_E dV=\int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 r(8-2r^2)\,dr\,d\theta$$

$$=\dots=16\pi.$$

$$\iiint_E z\,dV=\int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} z\,r\,dz\,dr\,d\theta=$$

$$= \frac{\iiint_E z\, dV}{\iiint_E dV}$$

$$E = \{(x,y,z) \mid x^2+y^2 \le z \le 8 - x^2 - y^2\}$$
$$= \{(r,\theta,z) \mid r \le 2,\ r^2 \le z \le 8 - r^2\}.$$

$$\iiint_E dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r(8-2r^2)\, dr\, d\theta$$

$$= \dots = 16\pi.$$

$$\iiint_E z\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} z\, r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^2 \frac{r}{2}\left((8-r^2)^2 - r^4\right) dr\, d\theta = \int_0^{2\pi}\int_0^2 \frac{1}{4}\left((8-r^2)^2 - r^4\right) d(r^2)\, d\theta$$

$$= \int_0^{2\pi}\int_0^4 \frac{1}{4}\left((8-t)^2 - t^2\right) dt\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\int_0^4 (64 - 16t)\, dt\, d\theta = \frac{1}{4}\int_0^{2\pi} \left[64t - 8t^2\right]_{t=0}^{4} d\theta$$

$$= \frac{1}{4}(64\cdot 4 - 8\cdot 16)\cdot 2\pi = \frac{16\cdot 8\cdot 2\pi}{4} = 64\pi.$$

$$z_0 = \frac{64\pi}{16\pi} = 4. \quad \square$$

(6) Compute the line integrals

TS

(6) Compute the line integrals over C, where C is the line segment from (3,1,2) to (1,2,5).

a) $\int_C y^2 z \, ds$

b) $\int_C z\,dx + x\,dy + x^2\,dz$

Solution a) Parametrization:

$$\vec{r}(t) = (1-t)\langle 3,1,2\rangle + t\langle 1,2,5\rangle$$

$$= \langle 3-3t+t, \; 1-t+2t, \; 2-2t+5t\rangle$$

$$= \langle 3-2t, \; 1+t, \; 2+3t\rangle \qquad 0 \le t \le 1.$$

$$\vec{r}'(t) = \langle -2, 1, 3\rangle, \qquad |\vec{r}'(t)| = \sqrt{2^2+1^2+3^2} = \sqrt{14}$$

$$\int_C y^2 z\, ds = \int_0^1 (1+t)^2 (2+3t)\, |\vec{r}'(t)|\, dt$$

$$= \sqrt{14}\int_0^1 (1+t)^2(2+3t)\,dt = \sqrt{14}\int_0^1 (1+2t+t^2)(2+3t)\,dt$$

$$= \sqrt{14}\int_0^1 (2+4t+2t^2+3t+6t^2+3t^3)\,dt$$

a) $\int_C y^2 z \, ds$

b) $\int_C z\,dx + x\,dy + x^2\,dz$

Solution a) Parametrization:

$$\bar{r}(t) = (1-t)\langle 3,1,2\rangle + t\langle 1,2,5\rangle$$

$$= \langle 3-3t+t,\ 1-t+2t,\ 2-2t+5t\rangle$$

$$= \langle \underbrace{3-2t}, \underbrace{1+t}, \underbrace{2+3t}\rangle \qquad 0 \le t \le 1.$$

$$\bar{r}'(t) = \langle -2, 1, 3\rangle, \quad |\bar{r}'(t)| = \sqrt{2^2+1^2+3^2} = \sqrt{14}$$

$$\int_C y^2 z\, ds = \int_0^1 (1+t)^2 (2+3t)\, |\bar{r}'(t)|\, dt$$

$$= \sqrt{14}\int_0^1 (1+t)^2(2+3t)\,dt = \sqrt{14}\int_0^1 (1+2t+t^2)(2+3t)\,dt$$

$$= \sqrt{14}\int_0^1 (2+4t+2t^2+3t+6t^2+3t^3)\,dt$$

$$= \sqrt{14}\int_0^1 (2+7t+8t^2+3t^3)\,dt$$

$$= \sqrt{14}\left[2t+\frac{7t^2}{2}+\frac{8t^3}{3}+\frac{3t^4}{4}\right]_{t=0}^{t=1}$$

$$= \sqrt{14}\left(2+\frac{7}{2}+\frac{8}{3}+\frac{3}{4}\right) \qquad \square$$

b) $\int_C z\,dx + x\,dy + x^2\,dz = \int_0^1 z(t)x'(t)\,dt$

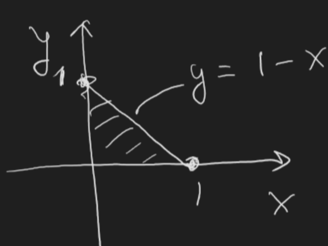
$+ x(t)y'(t)\,dt + x(t)^2 z'(t)\,dt$

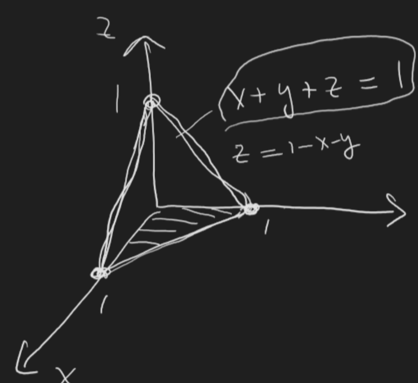
$= \int_0^1 \left((2+3t)(-2) + 3-2t + (3-2t)^2\cdot 3\right)\,dt$

$= \dots = 8.$ □

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b \overbrace{f(x(t),y(t),z(t))}^{f(\bar r(t))}\,|\bar r'(t)|\,dt}$$

7. Compute the flux of the vector field $\bar F(x,y,z) = \langle \overbrace{z+y}^{P}, \overbrace{x}^{Q}, \overbrace{z}^{R} \rangle$ across the surface of the tetrahedron enclosed by the coordinate planes and the plane $x+y+z=1$ given with the positive orientation





Solution

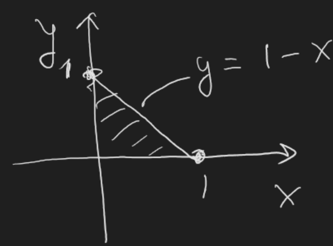
$\iint_S \bar F\cdot d\bar S$

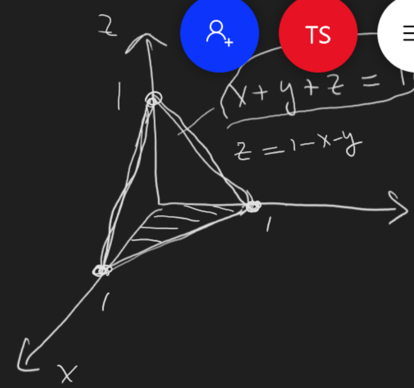
Div. Th.

$\iiint_E \operatorname{div}\bar F\,dV$

$\operatorname{div}\bar F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0+0+1=1$

positive orientation





Solution

$$\iint_S \bar{F} \cdot d\bar{S}$$

Div.Th.

$$\iiint_E \operatorname{div} \bar{F} \, dV$$

$$= \iiint_E dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left(1 - x - x(1-x) - \frac{(1-x)^2}{2} \right) dx$$

$$= \dots = \frac{1}{6}$$

$$\operatorname{div} \bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 0 + 1 = 1$$

$$E = \{ (x,y,z) \mid 0 \le z \le 1-x-y, \; 0 \le y \le 1-x, \; 0 \le x \le 1 \}$$

□