LECTURE 2: CONVEXITY

CONVEX SETS

DEF: THE SET $S \in \mathbb{R}^n$ IS CONVEX IF

CONVEX COMBINATION

$$\left.\begin{array}{c} x^1, x^2 \in S \\ \lambda \in (0,1) \end{array}\right\} \Rightarrow \lambda x^1 + (1-\lambda) x^2 \in S$$

CONVEX

NON-CONVEX

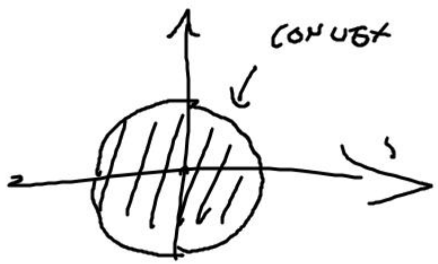
$\begin{bmatrix} x \\ y \end{bmatrix}$ X

<u>EX:</u>

- THE EMPTY SET ($\emptyset$) IS A CONVEX SET

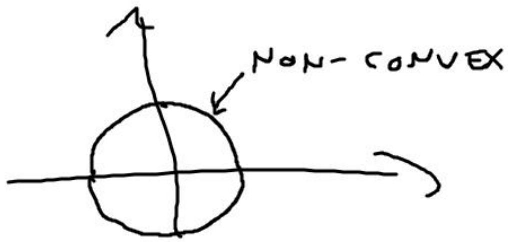
a) - THE SET $\{x \in \mathbb{R}^n \mid \|x\| \leq \alpha\}$ IS CONVEX FOR ANY $\alpha \in \mathbb{R}$

b) - $\{x \in \mathbb{R}^n \mid \|x\| = \alpha\}$ IS NON-CONVEX SET FOR ANY $\alpha > 0$



$\mathbb{R}^2$

a)



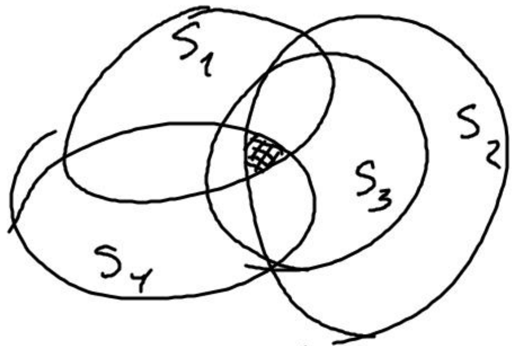
b)

- $\{1, 2, 3\}$ IS NON-CONVEX

PROPOSITION

LET S_k, $k \in K$, WHERE $S_k \subseteq \mathbb{R}^n$ IS CONVEX,
THEN THE INTERSECTION OF THE SETS
IS ALSO A CONVEX SET, i.e.

$$S = \bigcap_{k \in K} S_k \quad \text{IS CONVEX}$$



PROOF:

LET $\underline{x^1, x^2 \in S}$, $\underline{\lambda \in (0,1)}$.

$\Rightarrow$ $x^1 \in S_k$, $\forall k \in K$

$x^2 \in S_k$, $\forall k \in K$

$\lambda x^1 + (1-\lambda) x^2 \in S_k$, $\forall k \in K$ SINCE S_k IS CONVEX

$\Rightarrow$ $\underline{\lambda x^1 + (1-\lambda) x^2} \in \bigcap_{k \in K} S_k = S$

$\Rightarrow$ S IS CONVEX

- THE AFFINE HULL OF FINITELY MANY POINTS $w^1, \dots, w^k$ IS

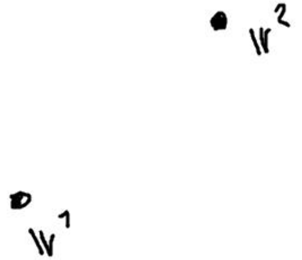
$$\mathrm{aff}(\{w^1,\dots,w^k\}) = \left\{ \lambda_1 w^1 + \dots + \lambda_k w^k \;\middle|\; \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

- THE CONVEX HULL ——— " ———

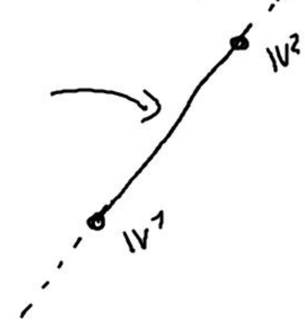
$$\mathrm{conv}(\{w^1,\dots,w^k\}) = \left\{ \lambda_1 w^1 + \dots + \lambda_k w^k \;\middle|\; \sum_{i=1}^{k} \lambda_i = 1 \;,\; \lambda_i \geq 0 \;,\; i = 1,\dots,k \right\}$$

$\mathrm{conv}(\{w^1, w^2, w^3\})$

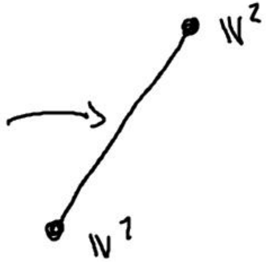
$\{w^1, w^2\}$



$\mathrm{aff}(\{w^1, w^2\})$

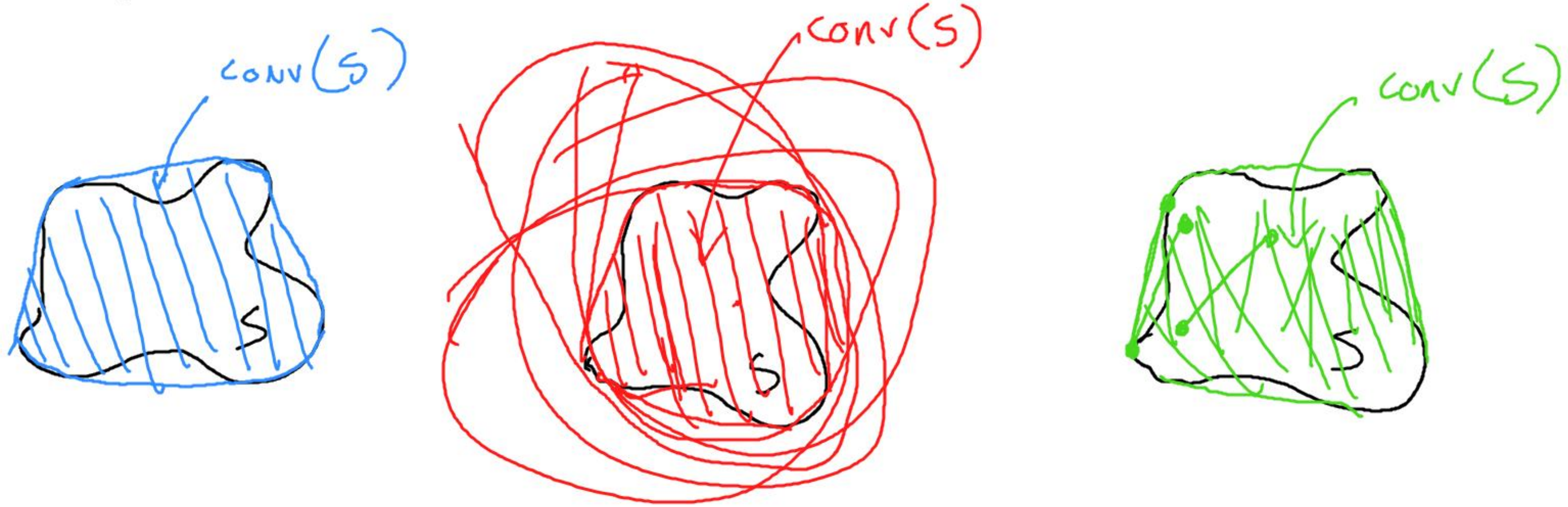


$\mathrm{conv}(\{w^1, w^2\})$



IN GENERAL, THE CONVEX HULL OF A SET S IS DEFINED AS

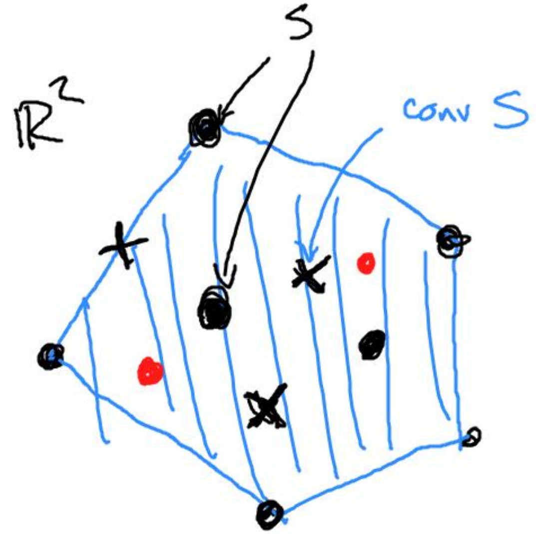
- a) THE UNIQUE MINIMAL CONVEX SET CONTAINING S
- b) THE INTERSECTION OF ALL CONVEX SETS CONTAINING S
- c) THE SET OF ALL CONVEX COMBINATIONS OF POINTS IN S



FROM c) EVERY POINT $x \in \text{conv}(S)$ CAN BE WRITTEN AS A CONVEX COMBINATION OF POINTS IN S.

<u>THEROM</u>: (CARATHEODORY'S THEOREM).

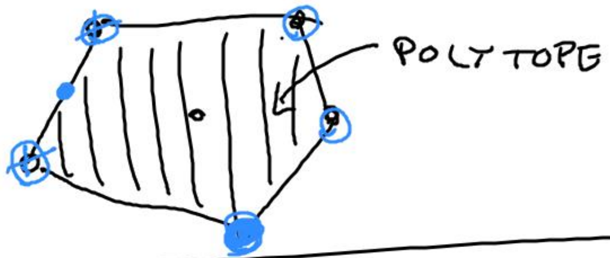
LET $x \in \text{conv}(S)$, WHERE $S \subseteq \mathbb{R}^n$, THEN x CAN BE WRITTEN AS A CONVEX COMB OF $n+1$ OR FEWER POINTS

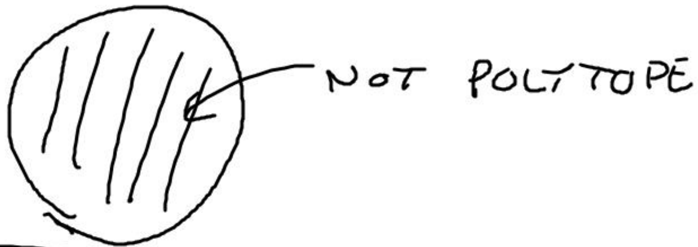


<u>PROOF</u>: SEE COURSE BOOK

DEF: A SET $P \subseteq \mathbb{R}^n$ IS A <u>POLYTOPE</u> IF IT IS THE CONVEX HULL OF FINITELY MANY POINTS IN $\mathbb{R}^n$.

EX.

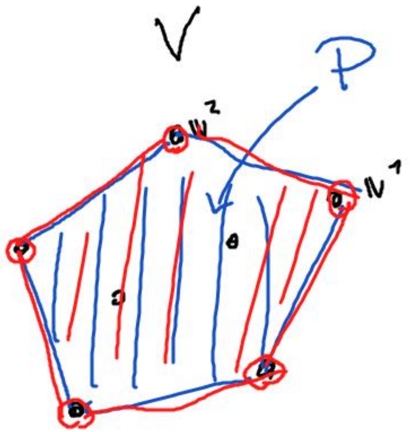




DEF: A POINT $\mathbb{v}$ OF A CONVEX SET P IS AN <u>EXTREME POINT</u> IF

$$\left.\begin{array}{r} \mathbb{v} = \lambda \mathbb{x}^1 + (1-\lambda)\mathbb{x}^2 \\ \mathbb{x}^1, \mathbb{x}^2 \in P \\ \lambda \in (0,1) \end{array}\right\} \Longrightarrow \mathbb{v} = \mathbb{x}^1 = \mathbb{x}^2.$$

THM: LET P BE THE POLYTOPE $\operatorname{conv}(V)$, WHERE $V = \{v^1, \dots, v^k\}$. THEN P IS EQUAL TO THE CONVEX HULL OF ITS EXTREME POINTS

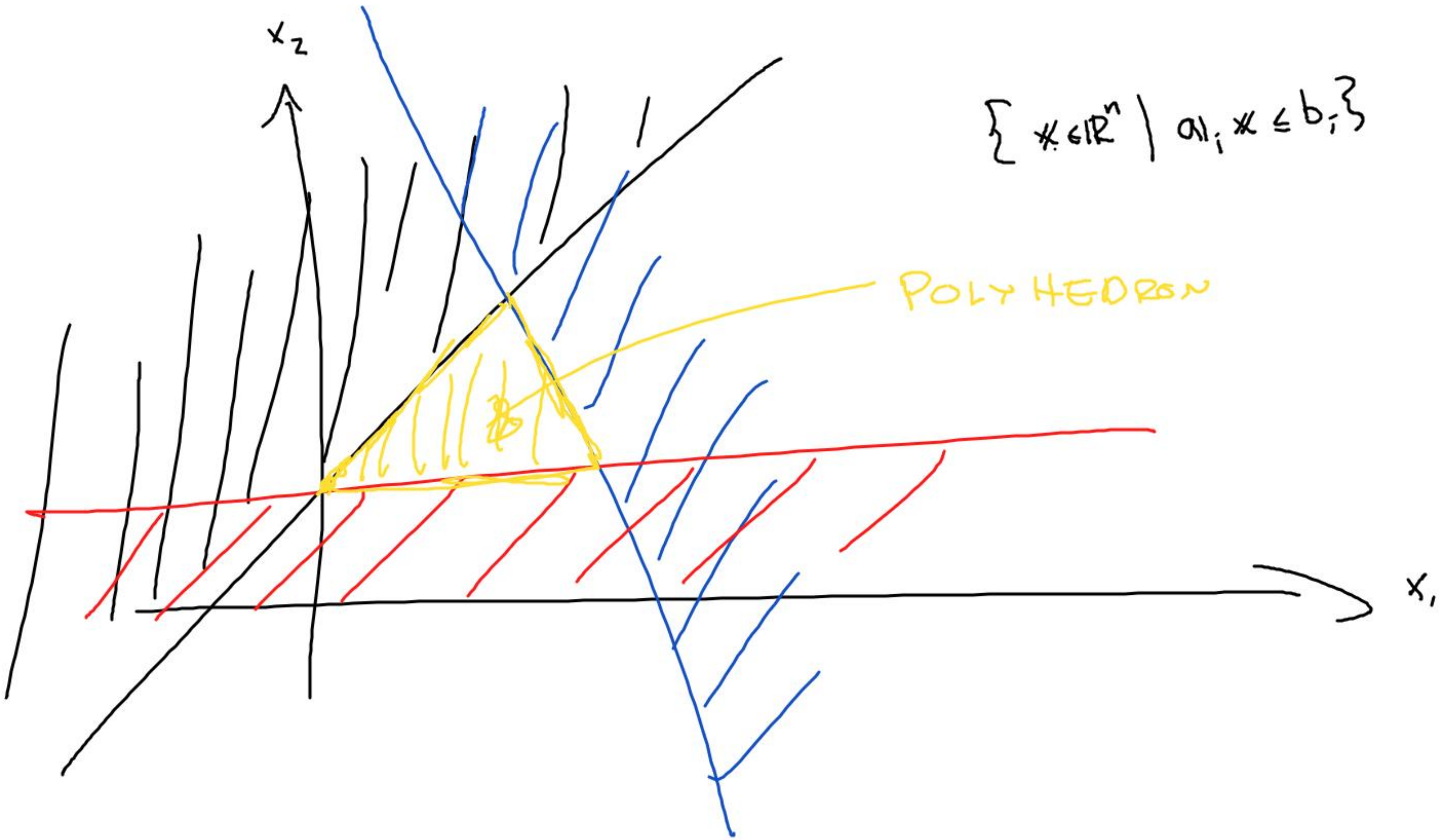


$\Longrightarrow$ EXTREME POINTS ARE INTERESTING

DEF: A SET $P \in \mathbb{R}^n$ IS A **POLYHEDRON** IF THERE EXISTS A MATRIX $A \in \mathbb{R}^{m \times n}$ AND A VECTOR $b \in \mathbb{R}^m$ SUCH THAT

$$P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$$

- $Ax \leq b \iff a_i x \leq b_i \quad , \; i = 1, \ldots, m$
- $\{ x \in \mathbb{R}^n \mid a_i x \leq b_i \}$ IS A **HALF-SPACE**
- $\Rightarrow$ P IS THE **INTERSECTION OF HALF-SPACES**



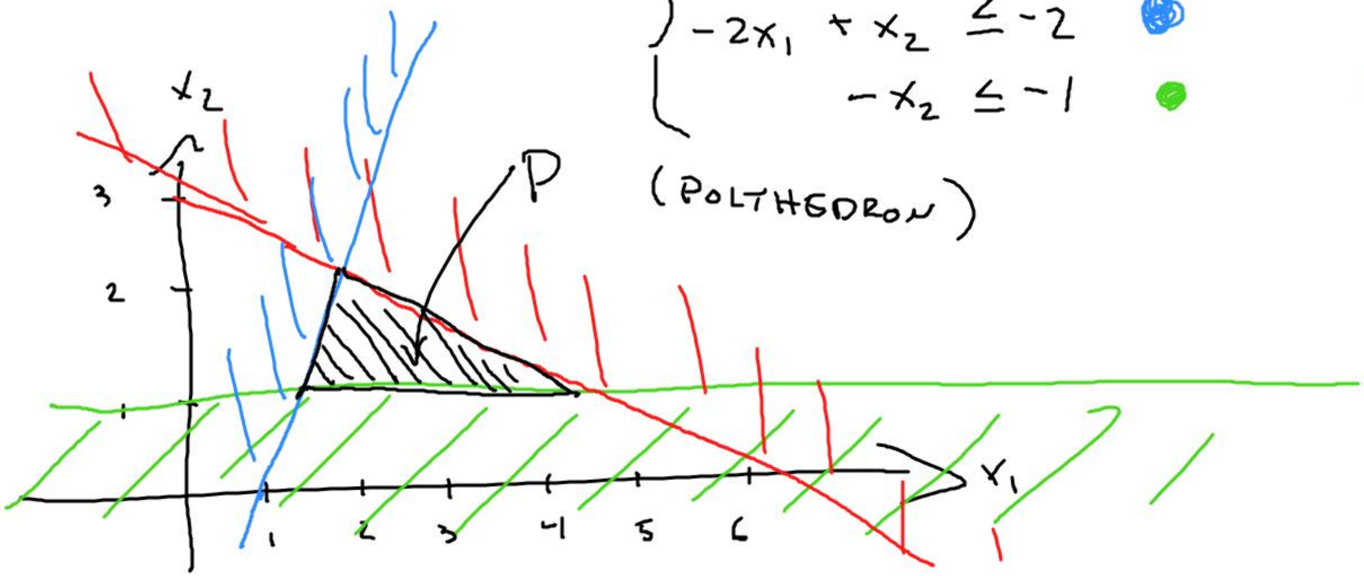
$$\{ x \in \mathbb{R}^n \mid a_i x \le b_i \}$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -1 \end{bmatrix}$, $\mathbb{b} = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}$ $\Rightarrow$ $A\mathbb{x} \leq \mathbb{b}$

$$\Leftrightarrow \begin{cases} x_1 + 2x_2 \leq 6 \\ -2x_1 + x_2 \leq -2 \\ -x_2 \leq -1 \end{cases}$$

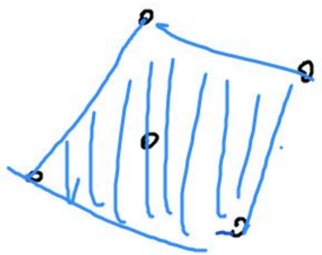
$x_2 \geq 1$

(POLYHEDRON)



POLYTOPE

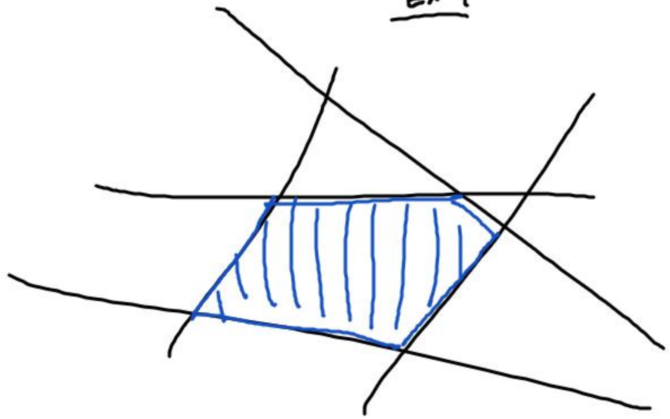
CONVEX HULL OF
FINITELY MANY POINTS



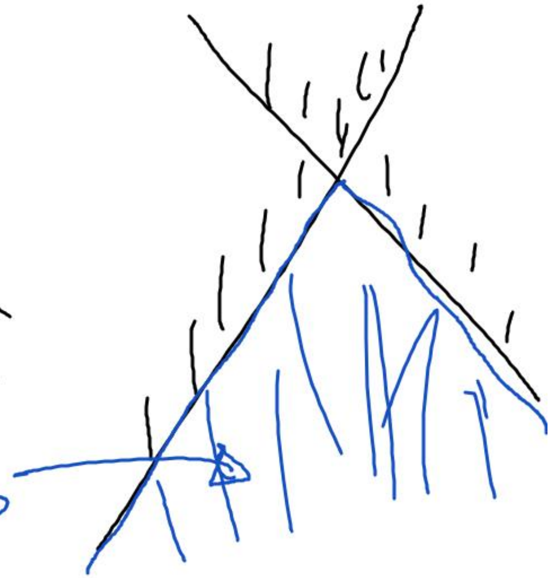
POLYHEDRON

INTERSECTION OF
FINITELY MANY HALF-SPACES

Ex 1



Ex 2



UNBOUNDED
SET

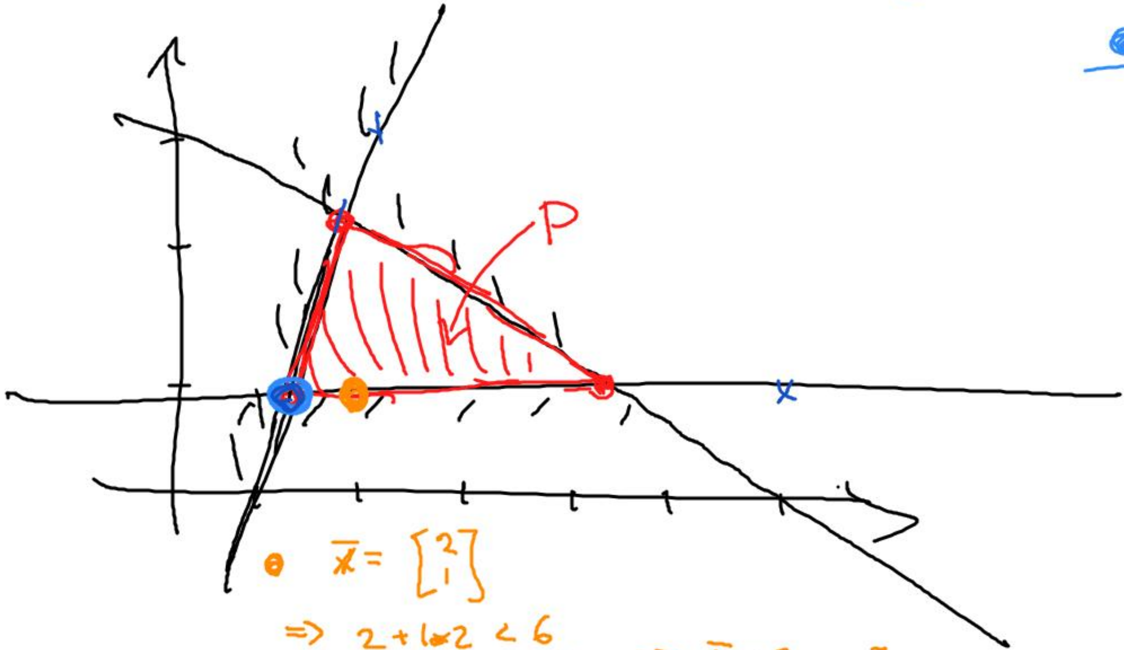
THEOREM LET $\bar{x} \in P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ (P IS A POLYHEDRON)
AND $\operatorname{rank} A = n$ AND $b \in \mathbb{R}^m$.
FURTHER, LET $\bar{A}x = \bar{b}$ BE THE EQUALITY SUBSYSTEM OF $Ax \le b$. THEN $\bar{x}$ IS AN EXTREME POINT OF P IF AND ONLY IF $\operatorname{rank} \bar{A} = n$.

$\bar{A}$ CONTAINS ALL ROWS IN A WHERE $(A\bar{x})_i = b_i$

$$A = \begin{bmatrix} \cancel{2} & \cancel{1} \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \cancel{4} \\ 2 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$A\bar{x} = \begin{bmatrix} \cancel{2} \\ 2 \end{bmatrix} \le \begin{bmatrix} \cancel{4} \\ 2 \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \end{bmatrix}$$

EX. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}$. $\Rightarrow$
$$x_1 + 2x_2 \le 6$$
$$-2x_1 + x_2 \le -2 \quad \times$$
$$-x_2 \le -1 \quad \times$$



$\bar{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$\Rightarrow$
$$\left(\tfrac{3}{2}\right) + 2 \cdot 1 < 6$$
$$-2\left(\tfrac{3}{2}\right) + 1 = -2 \leftarrow$$
$$-1 = -1 \leftarrow$$

$\Rightarrow \bar{A} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$\text{rank } \bar{A} = 2 = n$

$\Longrightarrow$ $\bar{x}$ IS AN EXTREME POINT!

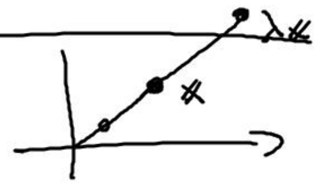
$\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

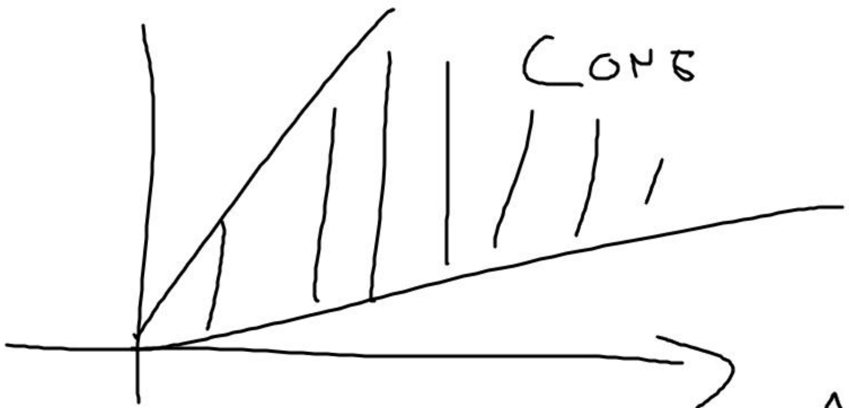
$\Rightarrow$
$$2 + 1 \cdot 2 < 6$$
$$-2 \cdot 2 + 1 < -2$$
$$-1 = -1 \leftarrow$$

$\Rightarrow \bar{A} = \begin{bmatrix} 0 & -1 \end{bmatrix}$

$\text{rank } \bar{A} = 1 < n$ $\Rightarrow$ $\bar{x}$ NOT AN EXTREME POINT

DEF: A set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda x \in C$ whenever $x \in C$ and $\lambda > 0$.





NOW AN IMPORTANT THM THAT SAYS:

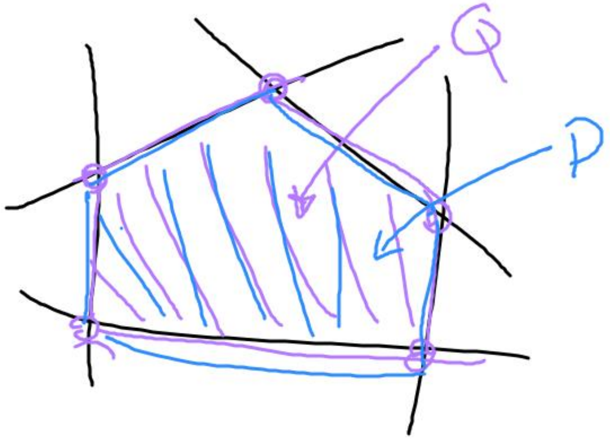
POLYHEDRON = POLYTOPE + CONE

THM: (REPRESENTATION THEOREM)

LET $Q = \{x \in \mathbb{R}^n \mid Ax \le b\}$ BE A ~~POLYTOPE~~ POLYHEDRON AND $\{w^1, \dots, w^k\}$ ARE ITS EXTREME POINTS. DEFINE $P := \operatorname{conv}(\{w^1, \dots, w^k\})$ AND $C := \{x \in \mathbb{R}^n \mid Ax \le 0\}$. THEN

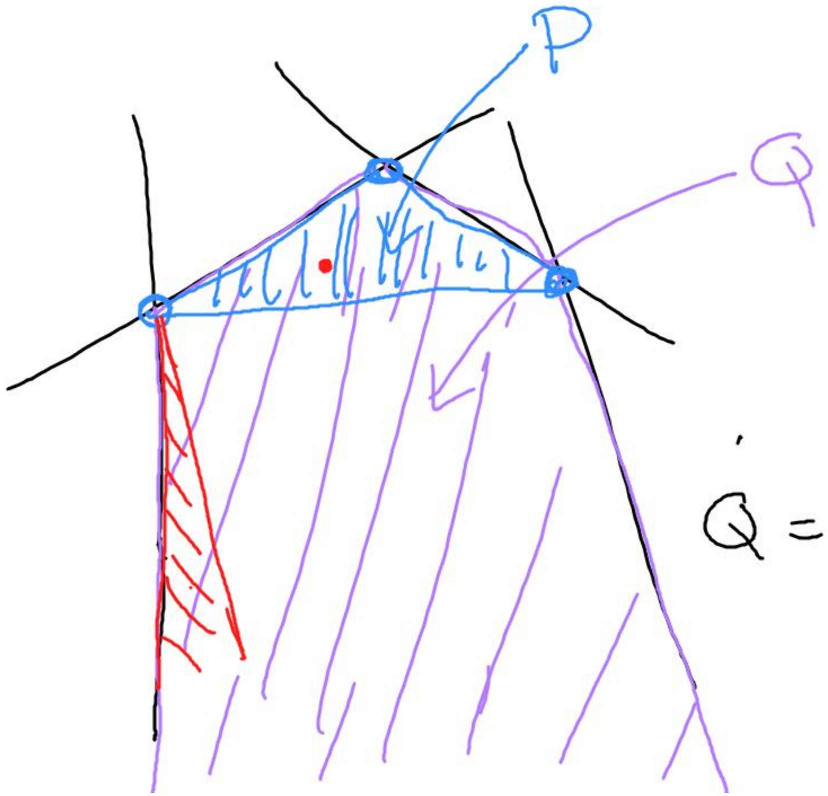
$$\underbrace{Q}_{\text{POLYHEDRON}} = \underbrace{P}_{\text{POLYTOPE}} + \underbrace{C}_{\text{CONE}} = \{x \in \mathbb{R}^n \mid x = u + v \text{ FOR SOME } u \in P,\ v \in C\}$$

BOUNDED POLYHEDRON



$$Q = P + 0$$

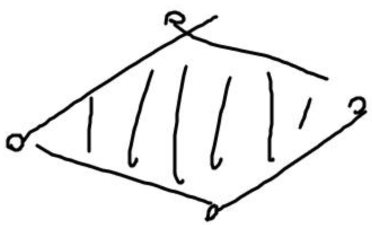
UNBOUNDED POLYHEDRON



$$Q = P + C$$

POLYTOPE

- EASY TO GET EXTREME POINTS
- "HARD" TO CHECK IF A NEW POINT IS IN THE POLYTOPE



POLYHEDRON

- HARD TO GET EXTREME POINTS
- EASY TO CHECK IF POINT IS INCLUDED