THM: (NECESSARY) $f \in C^1$

x^* IS A LOCAL MIN $\Longrightarrow \nabla f(x^*) = 0$

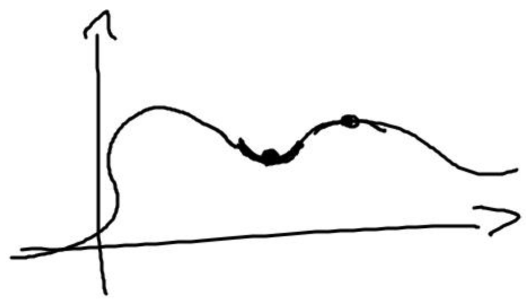
(CONSIDER $f(x) = x^3$, $x^* = 0$)

THM: (NECESSARY CONDITION)

IF $f \in C^2$ ON $\mathbb{R}^n$. THEN

x^* IS LOCAL MIN $\Longrightarrow \begin{cases} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) \succeq 0 \end{cases}$

$$(P) \quad \min f(x) \quad \text{s.t. } x \in \mathbb{R}^n$$



PROOF: 2nd ORDER TAYLOR EXPANSION AROUND x^* WHICH IS A LOCAL MIN IN THE DIRECTION p.

$$f(x^* + \alpha p) = f(x^*) + \underbrace{\alpha \nabla f(x^*)^T}_{=0} p + \frac{1}{2}\alpha^2 \underbrace{p^T \nabla^2 f(x^*) p}_{<0} + o(\alpha^2)$$

ASSUME x^* LOCAL MIN WITH $\nabla f(x^*)$ BUT WHER THERE EXISTS p: $\quad p^T \nabla f(x^*) p \nless 0 \quad$ (i.e. $\nabla^2 f(x^*) \nsucceq 0$)

$\Rightarrow$ FOR SMALL α

$$f(x^* + \alpha p) < f(x^*)$$

$\Rightarrow$ x^* CANNOT BE LOCAL MIN.

$\blacksquare$

THM: (SUFFICIENT CONDITION)

IF $f \in C^2$ ON $\mathbb{R}^n$, THEN

$$\left.\begin{cases} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) \succ 0 \end{cases}\right. \Longrightarrow x^* \text{ IS LOCAL MIN}$$

PROOF: SIMILAR TO PREVIOUS.

THM: (NECESSARY AND SUFFICIENT CONDITIONS)

$$\boxed{\text{IF } f \in C^1 \text{ IS A CONVEX FUNCTION, THEN} \quad x^* \text{ IS GLOBAL MIN} \iff \nabla f(x^*) = 0}$$

PROOF: $\Rightarrow$: ALREADY SHOWN.

$\Leftarrow$: f IS CONVEX, SO $\forall x \in \mathbb{R}^n$

$$f(x) \geq f(x^*) + \underbrace{\nabla f(x^*)^T}_{=0}(x - x^*) = f(x^*)$$

$\Rightarrow$ x^* GLOBAL MIN $\blacksquare$

OPTIMALITY CONDITIONS WHEN $S \subset \mathbb{R}^n$

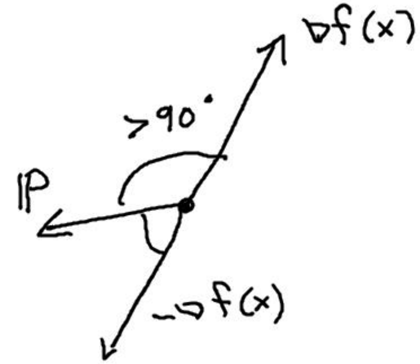
$$(P) \quad \min f(x) \quad \text{s.t. } x \in S$$

DEF: LET $x \in S$. A VECTOR p IS A <u>FEASIBLE DIRECTION</u> AT x IF

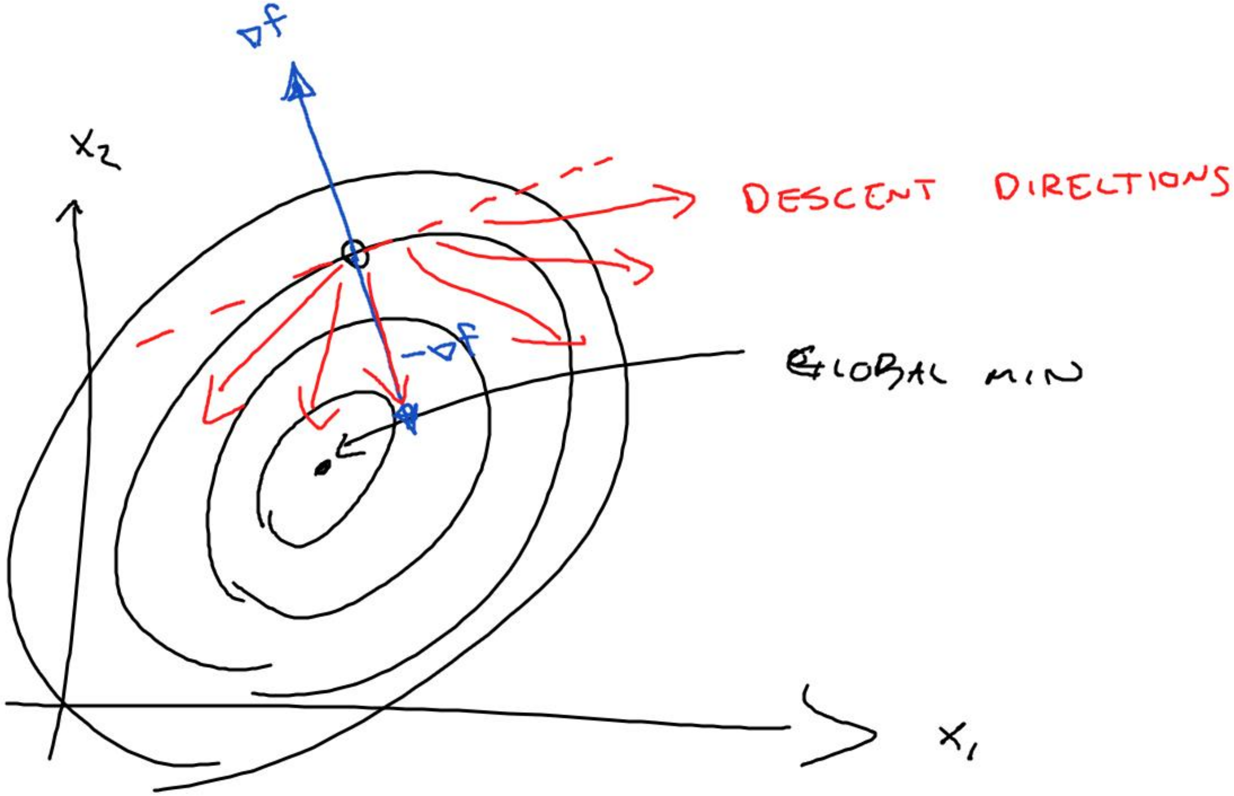
$$\exists\, \delta > 0 : \quad x + \alpha p \in S \quad \text{FOR ALL } \alpha \in [0, \delta]$$

DEF: LET $x \in \mathbb{R}^n$. A VECTOR p IS A <u>DESCENT DIRECTION</u> WITH RESPECT TO f AT x IF

$$\exists\, \delta > 0 : \quad f(x + \alpha p) < f(x)$$



<u>REMARK</u>: IF $f \in C^1$. IF $\nabla f(x)^T p < 0 \implies p$ IS A DESCENT DIRECTION



∇f
x_2
DESCENT DIRECTIONS
−∇f
GLOBAL MIN
x_1

EMILS OPTIMALITY CONDITION

$\boxed{(P) \quad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array}}$

x^* IS A LOCAL MIN TO (P) $\Longrightarrow$ "IT SHOULD NOT EXIST ANY VECTORS THAT ARE BOTH FEASIBLE DIRECTIONS AT x^* AND DESCENT DIRECTIONS"

PROOF: IF SUCH A VECTOR WOULD EXIST, JUST MOVE A SMALL AMOUND DIRECTION.

(NECESSARY)

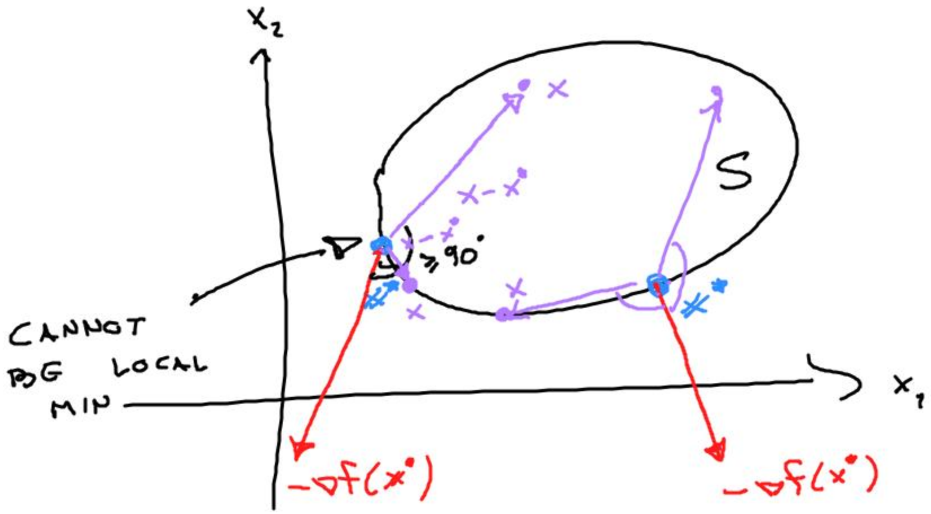
THM: SUPPOSE $S \subseteq \mathbb{R}^n$ AND $f \in C^1$ ON S.

a) $x^* \in S$ IS LOCAL MIN $\Rightarrow$ $\nabla f(x^*)^T p \geq 0$ FOR ALL FEASIBLE DIRECTIONS p AT x^*

b) SUPPOSE S IS CONVEX.

$x^* \in S$ IS LOCAL MIN $\Rightarrow$ $\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall\, x \in S$

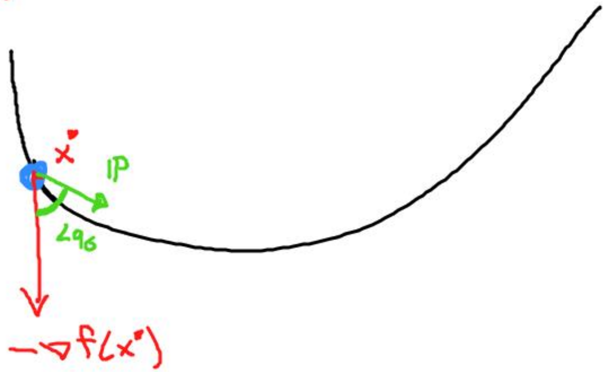
b)



S CONVEX

$$x^* \text{ LOCAL MIN} \Rightarrow \nabla f(x^*)^T(x - x^*) \geq 0 \quad \forall x \in S$$

$$\left(-\nabla f(x^*)^T(x - x^*) \leq 0 \right)$$



THM: x^* LOCAL MIN

$\Rightarrow \nabla f(x^*)^T p \geq 0$

FOR ALL FEASIBLE DIRECTIONS p.

PROOF: a)

TAYLOR EXPANSION.

$$f(x^* + \alpha p) = f(x^*) + \alpha \underbrace{\nabla f(x^*)^T p}_{<0} + o(\alpha)$$

ASSUME THAT p IS A FEASIBLE DIRECTION WITH $\nabla f(x^*)^T p < 0$

THEN FOR SMALL α

$$f(\underbrace{x^* + \alpha p}_{\in S}) < f(x^*)$$

CONTRADICTION THAT x^* IS LOCAL MIN

b) HOMEWORK.

THM: (NECESSARY & SUFFICIENT)

IF $S \subseteq \mathbb{R}^n$ IS CONVEX AND NONEMPTY AN $f \in C^1$ IS CONVEX, THEN

$$x^* \text{ GLOBAL MIN TO (P)} \iff \nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in S$$

REMARK: IF $S = \mathbb{R}^n$ THEN THE EXPESSION TO THE RIGHT IS EQUIVALENT WITH $\nabla f(x^*) = 0$

ASSUME THAT $S \subseteq \mathbb{R}^n$ IS CONVEX

A <u>STATIONARY POINT</u> x^* IS A POINT FULFILLING a) – d)

a) $\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in S$

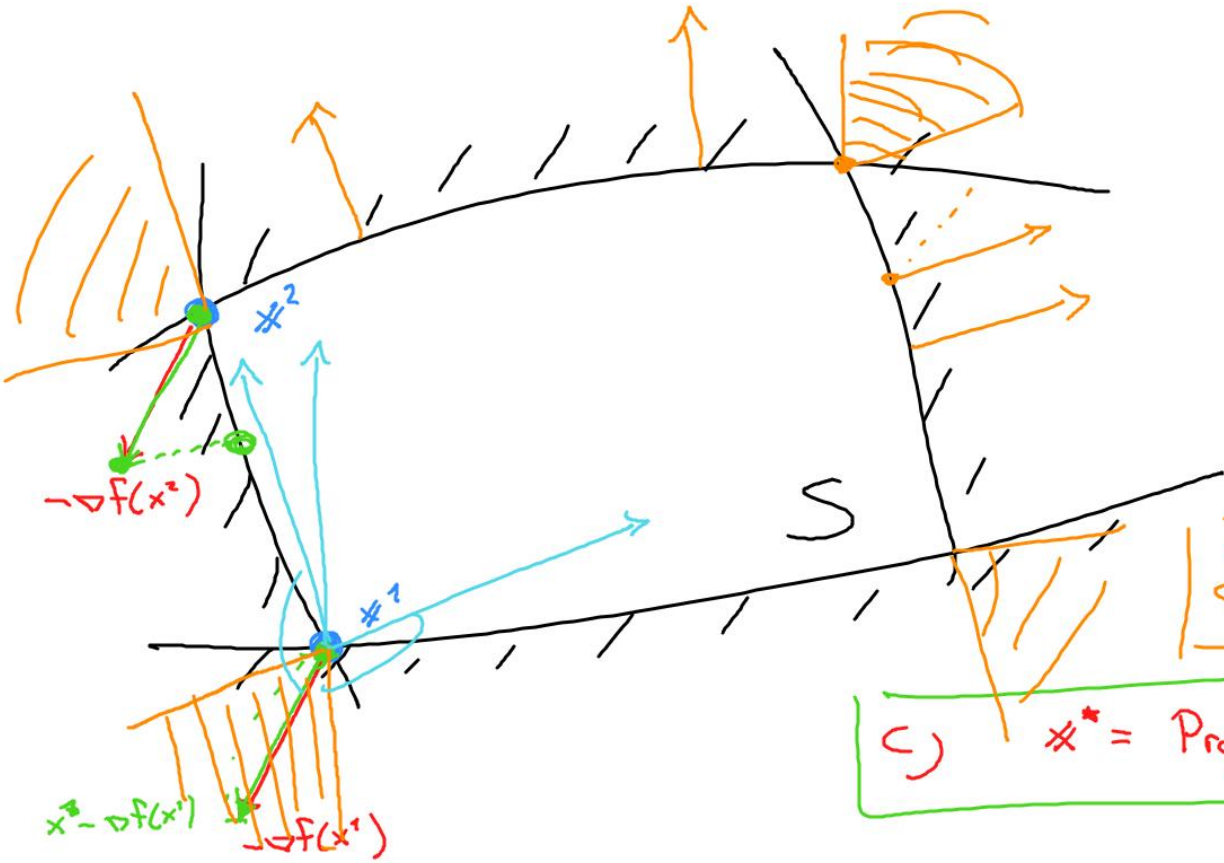
b) $\min_{x \in S} \nabla f(x^*)^T (x - x^*) = 0$

$$\boxed{\text{c)} \quad x^* = \mathrm{Proj}_S \left[x^* - \nabla f(x^*) \right]}$$

$$\boxed{\text{d)} \quad -\nabla f(x^*) \in N_S(x^*)}$$

<u>DEF</u>: SUPPOSE $S \subseteq \mathbb{R}^n$ IS CLOSED AND CONVEX. LET $x \in S$. THEN THE <u>NORMAL CONE</u> TO S AT x IS

$$N_S(x) = \left\{ p \in \mathbb{R}^n \,/\, p^T (y - x) \leq 0,\ y \in S \right\}$$



NORMAL CONE IS ALL VECTORS POINTING OUT FROM THESE SE

STRAIGHT

d $\boxed{-\nabla f(x^*) \in N_S(x^*)}$

c) $\boxed{x^* = \mathrm{Proj}_S\left[x^* - \nabla f(x^*)\right]}$

$\boxed{\begin{array}{l}\text{THM: (NECESSARY)} \\ \text{SUPPOSE } S \subseteq \mathbb{R}^n \text{ IS CONVEX. THEN} \\ x^* \text{ IS LOCAL MIN} \implies x^* \text{ IS STATIONARY}\end{array}}$

THM: (NECESSARY & SUFFICIENT)

SUPPOSE $S \subseteq \mathbb{R}^n$ IS CONVEX AND f IS CONVEX. THEN

x^* GLOBAL MIN $\iff$ x^* IS STATIONARY

NOTE: WE HAVE 4 VERSIONS OF STATIONARY POINTS WHEN $S \subseteq \mathbb{R}^n$ IS CONVEX. WHICH ONE CAN BE EXTENDED TO ALSO NON-CONVEX SETS?

$$\boxed{-\nabla f(x^*) \in N_S(x^*)}$$