LECTURE 5: OPTIMALITY CONDITION

"NECESSARY COND": x^* LOCAL MIN $\Rightarrow$ [SOMETHING HOLDS]

"SUFFICIENT COND": [SOMETHING HOLDS] $\Rightarrow$ x^* LOCAL MIN

- FROM LECTURE 3. $\boxed{\begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array}}$ (P)

- ASSUMED THAT $S \subseteq \mathbb{R}^n$ WAS A CONVEX SET

NEC. COND

x^* IS LOCAL MIN TO (P) $\Rightarrow$ x^* IS A STATIONARY POINT

STATIONARY POINT WHEN S IS CONVEX

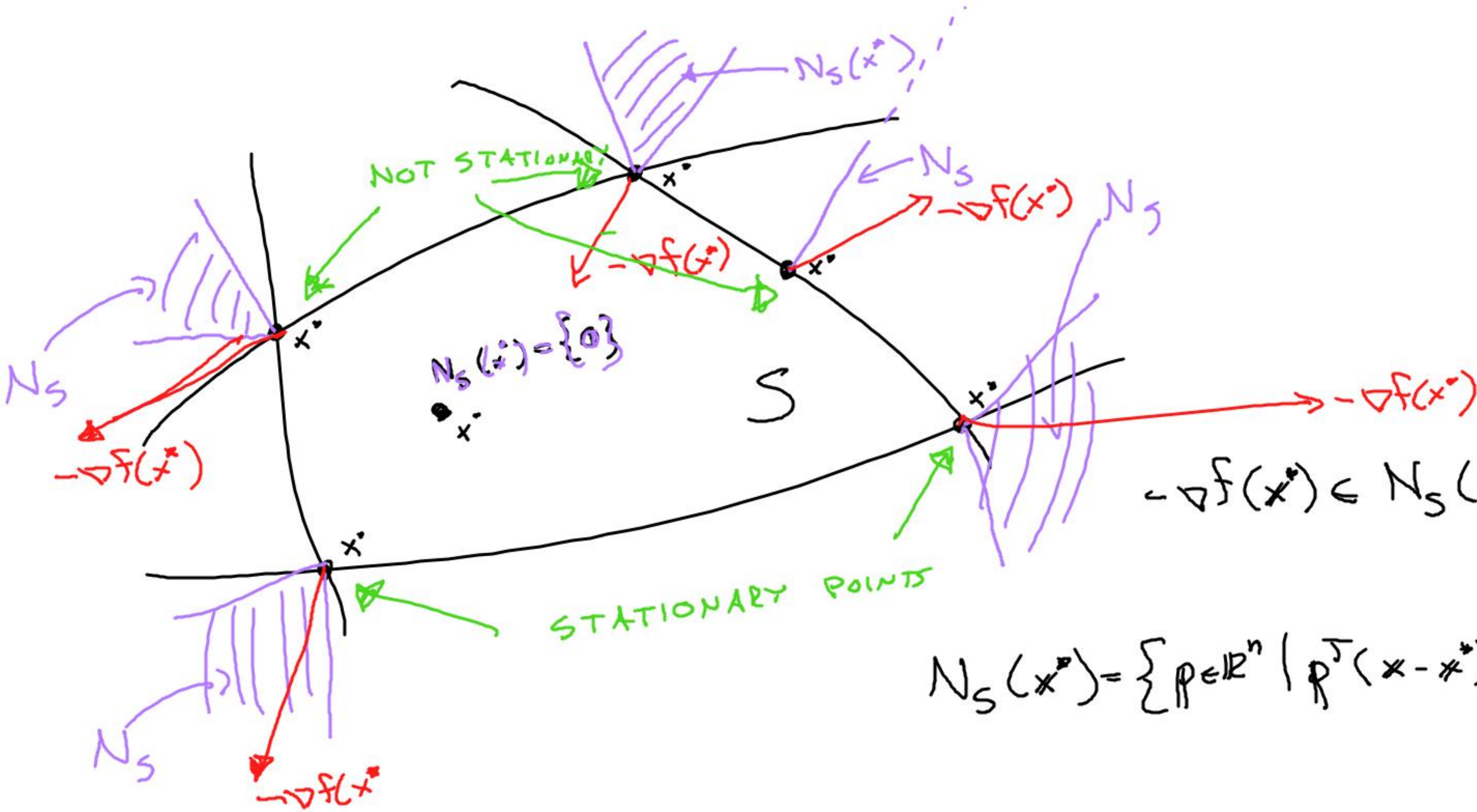
a) $\nabla f(x^*)^T(x - x^*) \geq 0 \quad , \quad \forall x \in S$

b) $\min_{x \in S} \nabla f(x^*)(x - x^*) = 0$

c) $x^* = \mathrm{Proj}_S(x^* - \nabla f(x^*))$

d) $\boxed{-\nabla f(x^*) \in N_S(x^*)}$

$$N_S(x^*) = \{ p \in \mathbb{R}^n \mid p^T(x - x^*) \leq 0 \ , \ \forall x \in S \}$$



$$-\nabla f(x^*) \in N_S(x^*)$$

$$N_S(x^*) = \{p \in \mathbb{R}^n \mid p^T(x - x^*) \le 0, \ \forall x \in S\}$$

EMILS OPTIMALITY CONDITION

$$\boxed{\begin{array}{l} \min f(x) \\ \text{s.t.} \quad x \in S \end{array}}$$

x^* IS A LOCAL MIN $\Rightarrow$ "IT SHOULD NOT BE POSSIBLE TO FIND A DIRECTION FROM x^* WHICH IS A FEASIBLE DIRECTION AND A DESCENT DIRECTION"

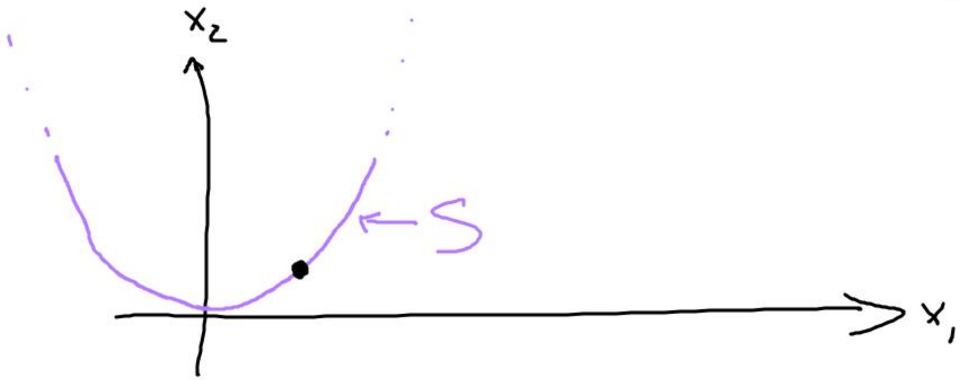
$$\boxed{\Rightarrow \underbrace{A(x^*)}_{\substack{\text{SET OF} \\ \text{FEASIBLE DIRECTIONS} \\ \text{AT } x^*}} \cap \underbrace{B(x^*)}_{\substack{\text{SET OF DESCENT} \\ \text{DIRECTIONS AT } x^*}} = \emptyset}$$

DEF: (CONE OF FEASIBLE DIRECTIONS)

LET $S \subseteq \mathbb{R}^n$ BE A NONEMPTY CLOSED SET. THE CONE OF FEASIBLE DIRECTIONS FOR S AT $\mathbf{x} \in S$ IS

$$R_S(\mathbf{x}) = \left\{ \mathbf{p} \in \mathbb{R}^n \mid \exists \delta > 0, \ \mathbf{x} + \alpha \mathbf{p} \in S, \ \forall \alpha \in [0, \delta] \right\}$$

EX: $S = \{ \mathbf{x} \in \mathbb{R}^2 \mid x_2 = x_1^2 \}$



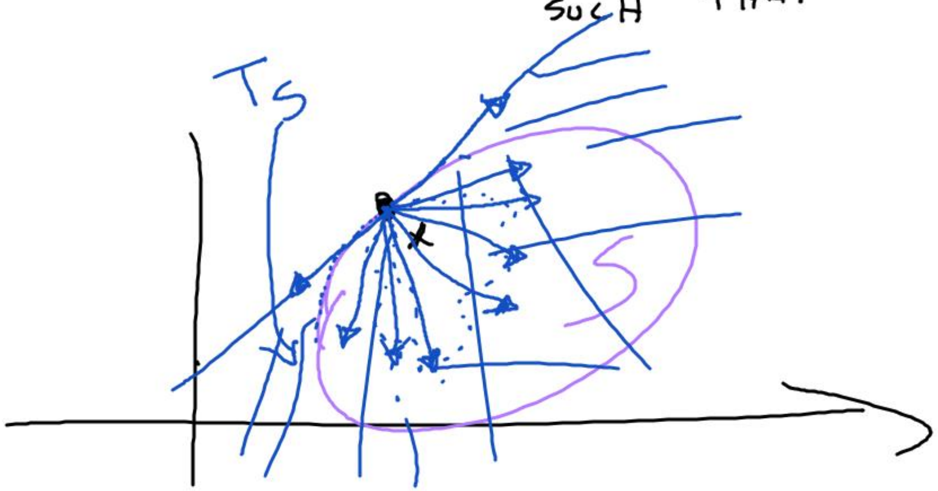
$R_S(\mathbf{x}) = \emptyset \quad \forall \ \mathbf{x} \in S$

DEF: (TANGENT CONE)

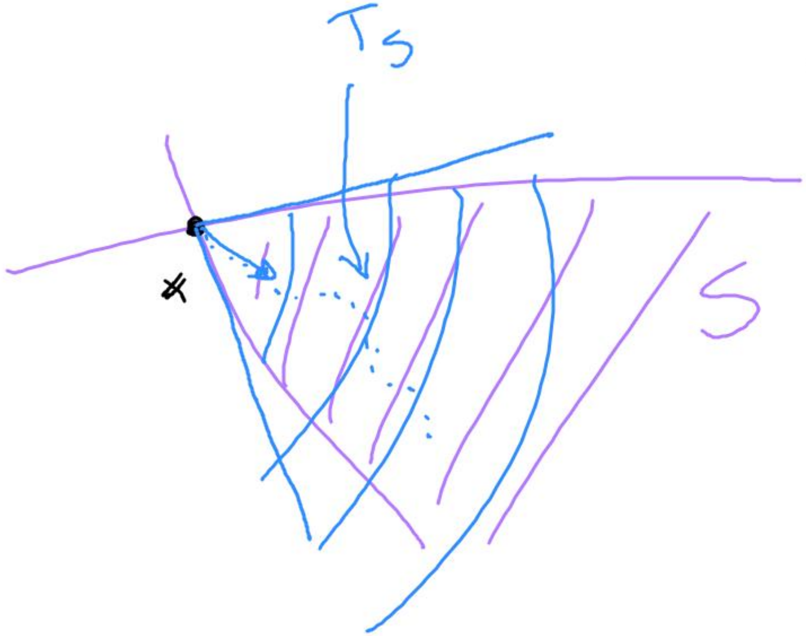
THE TANGENT CONE FOR S AT $x \in S$ IS

$$T_S(x) = \left\{ p \in \mathbb{R}^n \;\middle|\; \exists \{x_k\}_{k=0}^{\infty} \subset S,\ \{\lambda_k\}_{k=0}^{\infty} \subset (0,\infty), \text{ SUCH THAT } \lim_{k\to\infty} x_k = x,\ \lim \lambda_k (x_k - x) = p \right\}$$

EX. 1

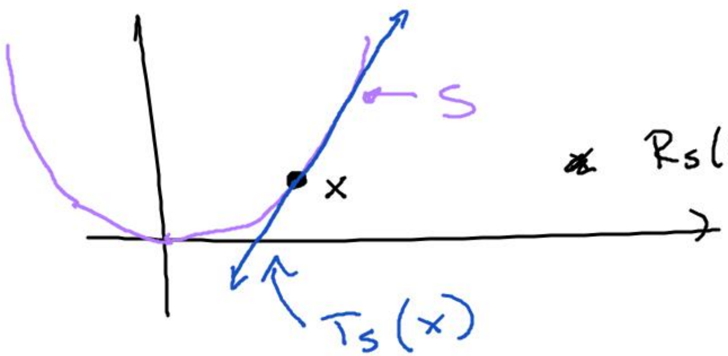


Ex2:



Ex.

$$S = \{ x \in \mathbb{R}^2 \mid x_2 = x_1^2 \}$$

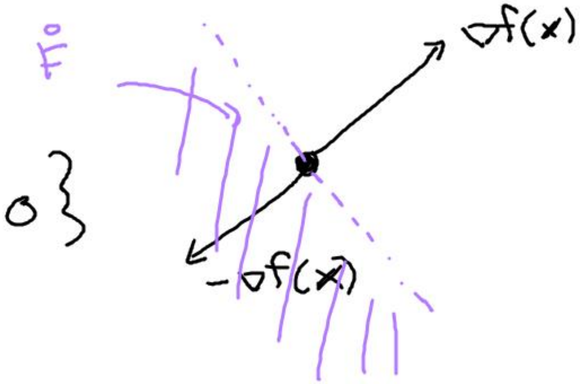


$R_S(x) = \emptyset$

$$\underline{R_S(x) \subseteq T_S(x)}$$

DEF: (CONE OF DESCENT DIRECTIONS)

$$\mathring{F}(x) = \{\, p \in \mathbb{R}^n \mid \nabla f(x)^T p < 0 \,\}$$



THM (GEOMETRIC OPTIMALITY CONDITIONS)

CONSIDER THE PROBLEM $\begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array}$ AND ASSUME $f \in C^1$. THEN

$$x^* \text{ IS LOCAL MIN} \implies \mathring{F}(x^*) \cap T_S(x^*) = \emptyset$$

PROOF: IN BOOK.

USEFUL CONDITIONS

$$\boxed{\begin{aligned} &\min \ f(x) \\ &\text{s.t.} \quad g_i(x) \le 0 \quad , i=1,\dots,m \end{aligned}}$$

WHERE $f: \mathbb{R}^n \to \mathbb{R} \in C^1$, $g_i: \mathbb{R}^n \to \mathbb{R} \in C^1$

DEF: $I(x) = \{ i \in \{1,\dots,m\} \mid g_i(x) = 0 \}$

SET OF ACTIVE CONSTRAINTS, i.e. ALL CONSTRAINTS THAT AR ACTIVE ($g_i(x)=0$) AT THE POINT x

EX: $g_1(x) = x_1^2 + x_2^2 - 1 \le 0$
$g_2(x) = x_2 \le 0$

$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $g_1(x) = -1 < 0$
$g_2(x) = 0 = 0$

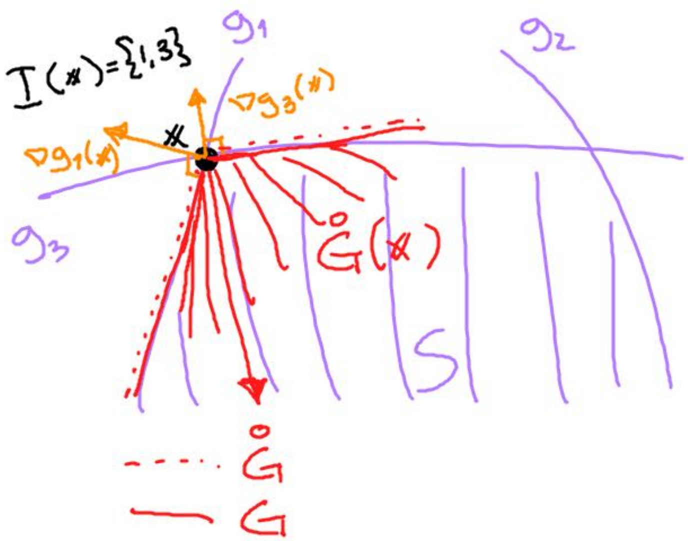
$\Rightarrow I(x) = \{2\}$

DEF: (GRADIENT CONES)

INNER GRADIENT CONE TO S AT $\bar{x}$: $\mathring{G}(\bar{x}) = \{ p \in \mathbb{R}^n \mid \nabla g_i(\bar{x})^T p < 0, \forall i \in I(\bar{x}) \}$

GRADIENT CONE TO S AT $\bar{x}$: $G(\bar{x}) = \{ p \in \mathbb{R}^n \mid \nabla g_i(\bar{x})^T p \leq 0, \forall i \in I(\bar{x}) \}$

EX:



THM:

$$\mathrm{cl}\ \mathring{G}(\#) \subseteq R_S(\#) \subseteq T_S(\#) \subseteq G(\#)$$

" $\dot{=}$ "

FOR NICELY BEHAVING SETS

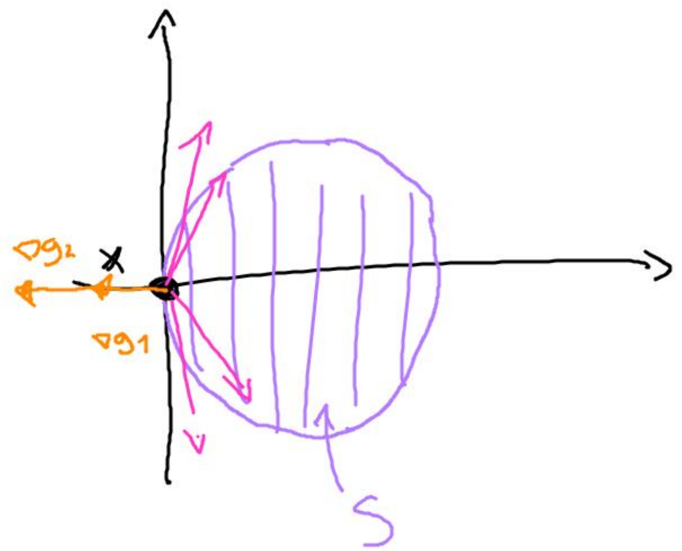
$S \in \mathbb{R}^2$

Ex 1:

$$S = \begin{cases} -x_1 \le 0 & g_1 \\ (x_1-1)^2 + x_2^2 \le 1 & g_2 \end{cases}$$

$$\nabla g_1(x^*) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla g_2(x^*) = \begin{bmatrix} 2(x_1-1) \\ 2x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \qquad I(x^*) = \{1,2\}$$



$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_S(x^*) = \{ p \in \mathbb{R}^2 \mid p_1 > 0 \}$$

$$T_S(x^*) = \{ p \in \mathbb{R}^2 \mid p_1 \geq 0 \}$$

$$\mathring{G}(x^*) = \{ p \in \mathbb{R}^2 \mid p_1 > 0 \}$$

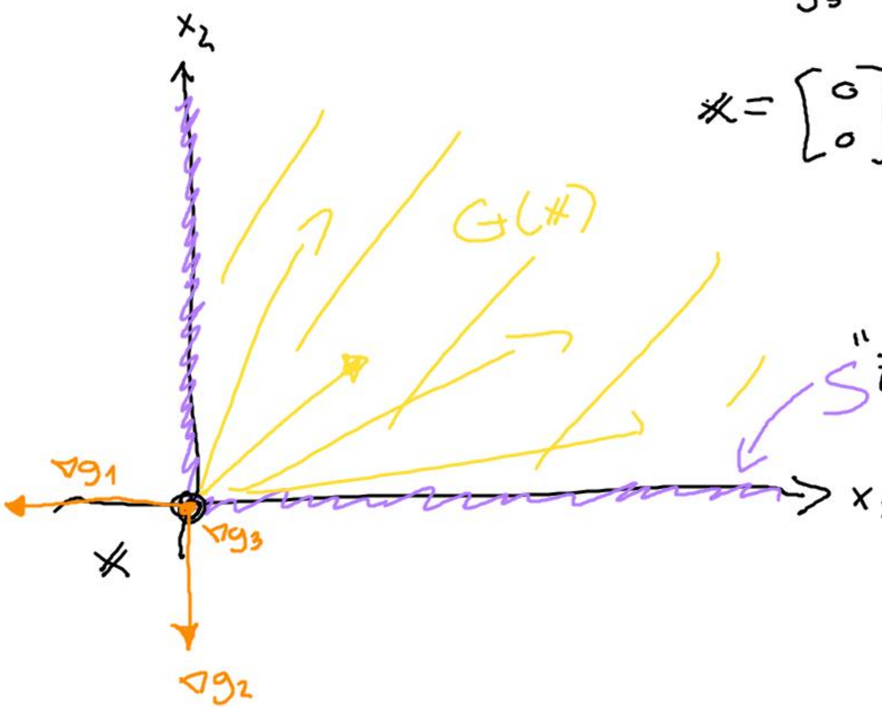
$$G(x^*) = \{ p \in \mathbb{R}^2 \mid p_1 \geq 0 \}$$

"="

Ex 2:

$$S = \begin{cases} -x_1 \le 0 & g_1 \\ -x_2 \le 0 & g_2 \\ x_1 x_2 \le 0 & g_3 \end{cases} \qquad \nabla g_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \nabla g_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \nabla g_3 = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$I(x^*) = \{1, 2, 3\}$$



$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_S(x^*) = S = \{ p \in \mathbb{R}^2 : p_1 \geq 0,\ p_2 \geq 0,\ p_1 \cdot p_2 = 0 \}$$

$$T_S(x^*) = S \quad \text{——} \quad " \quad \text{————}$$

"≠"

$$\mathring{G}(x^*) = \emptyset$$

$$G(x^*) = \{ p \in \mathbb{R}^2 \mid p_1 \geq 0,\ p_2 \geq 0 \}$$

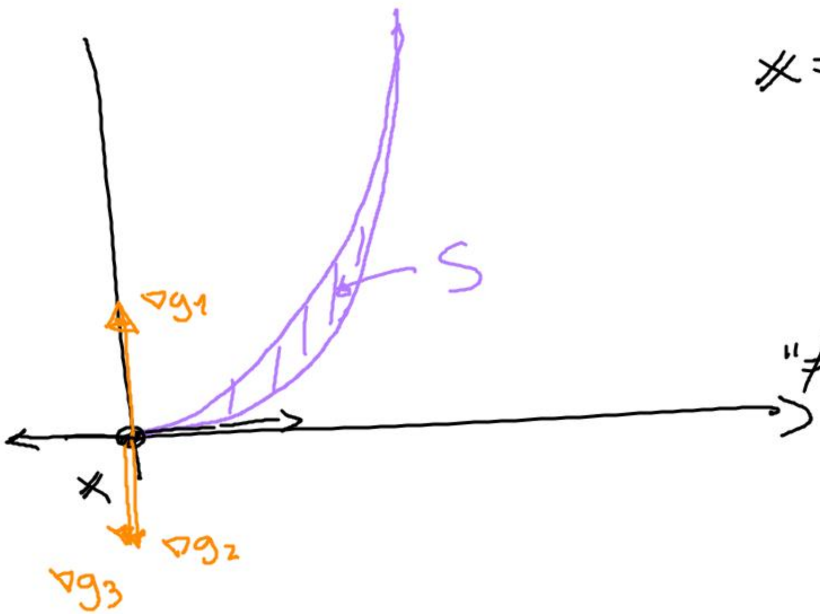
Ex 3:

$$S = \begin{cases} -x_1^3 + x_2 \leq 0 & g_1 \\ x_1^5 - x_2 \leq 0 & g_2 \\ -x_2 \leq 0 & g_3 \end{cases}$$

$$\nabla g_1 = \begin{bmatrix} -3x_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla g_2 = \begin{bmatrix} 5x_1^4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad I(x^*) = \{1,2,3\}$$

$$\nabla g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$



$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_S(x^*) = \emptyset$$

$$T_S(x^*) = \{ p \in \mathbb{R}^2 : p_1 \geq 0,\ p_2 = 0 \}$$

"$\neq$"

$$\mathring{G}(x^*) = \emptyset$$

$$G(x^*) = \{ p \in \mathbb{R}^2 \mid p_2 = 0 \}$$

FRITZ JOHN CONDITIONS

USE THAT $\mathring{G}(x) \subseteq T_S(x)$

$$\Rightarrow \quad \boxed{x^* \text{ LOCAL MIN} \Rightarrow \mathring{F}(x^*) \cap \mathring{G}(x^*) = \emptyset}$$

MUCH WEAKER RESULT, FOLLOWS DIRECTLY FROM GEOMETRIC OPT. COND. SINCE $\mathring{G}(x) \subseteq T_S(x)$.

$$\Leftrightarrow \quad x^* \text{ LOCAL MIN} \Rightarrow \begin{cases} \nabla f(x^*)^T p < 0 & \leftarrow \mathring{F} \\ \nabla g_i(x^*)^T p < 0, \quad i \in I(x) & \leftarrow \mathring{G} \end{cases}$$

DESCENT

FEASIBLE

MUST BE UNSOLVABLE FOR p

FARKAS' LEMMA

LET $A \in \mathbb{R}^{m \times n}$ AND $\mathbb{b} \in \mathbb{R}^{m}$.

THEN EXACTLY ONE OF THE FOLLOWING SYSTEMS IS SOLVABLE AND THE OTHER IS NOT.

$$\begin{cases} A\mathbb{x} = \mathbb{b} \\ \mathbb{x} \geq 0 \end{cases} \qquad\qquad \begin{cases} A^T\mathbb{y} \leq \mathbb{0} \\ \mathbb{b}^T\mathbb{y} > 0 \end{cases}$$

PROOF: COMES LATER

T

$$x^* \text{ LOCAL MIN} \Longrightarrow \begin{cases} \nabla f(x^*)^T p < 0 \\ \nabla g_i(x^*)^T p < 0, \quad i \in I(x^*) \end{cases}$$

IS UNSOLVABLE

[BY FARKAS' LEMMA]

$$\boxed{\Longrightarrow \begin{cases} \mu_0 \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0 \\ \mu_i g_i(x^*) = 0, \quad i = 1, \dots, m \\ \mu_0, \mu_i \geq 0, \quad i = 1, \dots, m \\ (\mu_0, \mu_1, \dots, \mu_m) \neq 0^{m+1} \end{cases} \text{ IS SOLVABLE FOR } (\mu_0, \mu_1, \dots, \mu_m)}$$

FRITZ JOHN CONDITION