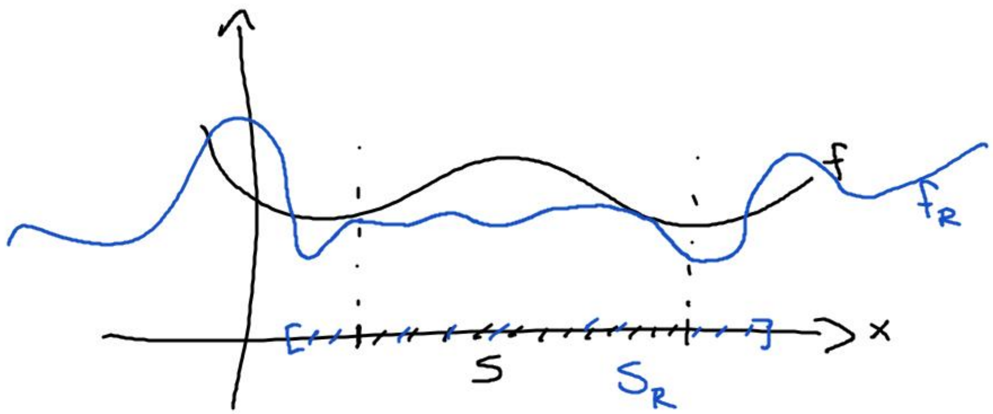
LECTURE 7: LAGRANGIAN DUALITY/RELAXATION

RELAXION METHODS

- RELAX THE PROBLEM YOU ARE CONSIDERING
- CREATE SOME "EASIER" PROBLEM THAN CAN HELP WITH THE ORIGINAL PROBLEM
- EX: REMOVE CONSTRAINTS

RELAXATIONS

$$(P) \quad f^* = \boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}}$$



WE SAY THAT A PROBLEM

$$(P') \quad f_R^* = \boxed{\begin{array}{ll} \min & f_R(x) \\ \text{s.t.} & x \in S_R \end{array}}$$

IS A RELAXATION TO (P) IF:

a) $S_R \supseteq S$ ← LARGER FEASIBLE SET

b) $f_R(x) \leq f(x)$ FOR ALL $\underline{x \in S}$ ← LOWER OBJECTIVE FUNCTION

THEOREM: [RELAXATION THEOREM]

CAN GET LOWER BOUNDS ON ORIGINAL PROBLEM

a) $f_R^* \leq f^*$

CAN REALIZE THAT OUR ORIG. PROBLEM IS INF.

b) IF THE RELAXED PROBLEM (P') IS INFEASIBLE, THEN THE ORIGINAL PROBLEM (P) IS ALSO INFEASIBLE.

c) IF (P') HAS AN OPTIMAL SOLUTION x_R^* FOR WHICH IT HOLDS THAT

$$x_R^* \in S \quad \text{AND} \quad f_R^*(x_R^*) = f(x_R^*),$$

THEN x_R^* IS ALSO OPTIMAL IN THE ORIGINAL PROBLEM (P)

EX: - REMOVE CONSTRAINTS
- SOLVE UNCONST. PROBLEM (P')
- CHECK IF SOLUTION SATISFIES ALL CONSTRAINTS
- IF SO, YOU ARE DONE

IF WE MANAGE TO SOLVE P', WE MIGHT GET LUCKY AND ALSO SOLVE P

PROOF:

$$f^* = \min_{\text{s.t. } x \in S} f(x) \quad , \quad f_R^* = \min_{\text{s.t. } x \in S_R} f_{R}(x)$$

a) $[f_R^* \leq f^*]$: CLEAR SINCE $f_R(x) \leq f(x) \ \forall x \in S$ AND $S \subseteq S_R$. $\square$

b) [IF P' INFEASIBLE $\Rightarrow$ P INFEASIBLE]: CLEAR SINCE $S_R \supseteq S$. AND IF $S_R = \emptyset \Rightarrow S = \emptyset$ $\square$

c) $\underline{f(x_R^*)} = f_R(x_R^*) \leq f_R(x) \leq \underline{f(x)}$ FOR ALL $\underline{x \in S}$

$\Rightarrow$ x_R^* MUST BE OPTIMAL SOLUTION TO ORIGINAL PROBLEM

$\square$

LAGRANGIAN RELAXATION

$$\boxed{\begin{aligned} f^* = \inf\ & f(x) \\ \text{s.t.}\ & g_i(x) \leq 0, \quad i=1,\dots,m \\ & x \in X \end{aligned}}$$

PRIMAL PROBLEM

IDEA IS TO REPLACE CONSTRAINTS WITH A PENALTY FOR BREAKING THEM

LAGRANGIAN DUAL FUNCTION:

$$\boxed{\begin{aligned} q(\mu) = \inf\ & \left[f(x) + \sum_{i=1}^{m} \mu_i g_i(x) \right] \\ \text{s.t.}\ & x \in X \end{aligned}}$$

NOTE: FOR $\mu_i \geq 0, i=1,\dots,m$, THIS PROBLEM IS A RELAXATION OF THE ORIGINAL / PRMAL PROBLEM

$$q(\mu) = \inf_{\text{s.t. } x \in X} \left[f(x) + \sum_{i=1}^{m} \mu_i g_i(x) \right] = \inf_{x \in X} \left[f(x) + \mu^T g(x) \right]$$

THEOREM: [WEAK DUALITY]

FOR ANY $\mu \geq 0$ AND ANY FEASIBLE x TO THE PRIMAL PROBLEM, IT HOLDS THAT

$$q(\mu) \leq f(x)$$

PROOF:

$$q(\mu) = \inf_{z \in X} \left[f(z) + \mu^T g(z) \right] \leq f(x) + \underbrace{\underbrace{\mu^T}_{\geq 0} \underbrace{g(x)}_{\leq 0}}_{\leq 0} \leq f(x) + 0 = f(x)$$

≤ 0 SINCE x FEASIBLE IN ORIGINAL PROBLEM $\square$

REMARK: IT ALSO STATES THAT $q(\mu) \leq f^*$ FOR ALL $\mu \geq 0$

GIVES LOWER BOUNDS ON OPTIMAL VALUE OF PRIMAL PROB

EX: $\boxed{f^* = \min_{\text{s.t. } x \geq 1} x^2}$

OPTIMAL SOLUTION: $x^* = 1$
OPTIMAL VALUE: $f^* = 1$

RELAX THE CONSTRAINT $x \geq 1$ $[1 - x \leq 0]$.

$$q(\mu) = \min_{\text{s.t. } x \in \mathbb{R}} \left[x^2 + \mu \cdot (1-x) \right] = \min_{x \in \mathbb{R}} \underbrace{\left(x - \frac{\mu}{2}\right)^2}_{=0} - \frac{\mu^2}{4} + \mu$$

FOR EACH $\mu \geq 0$, THIS IS EASILY SOLVED BY LETTING $x^* = \frac{\mu}{2}$

$$\Rightarrow \quad q(\mu) = -\frac{\mu^2}{4} + \mu$$

LOWER BOUNDS ON PRIMAL PROBLEM:

$\mu = 0$: $q(0) = 0$ $\Rightarrow$ $f^* \geq 0$

$\mu = 1$: $q(1) = \frac{3}{4}$ $\Rightarrow$ $f^* \geq 3/4$ BETTER

IDEA: - $g(\mu)$ ALWAYS GIVES LOWER BOUNDS TO f^* WHEN $\mu \geq 0$

- LETS TRY TO FIND THE BEST LOWER BOUND.

LAGRANGIAN DUAL PROBLEM

$$\boxed{\begin{aligned} g^* = \sup\ & g(\mu) \\ \text{s.t. } & \mu \geq 0 \end{aligned}}$$

FINDING THE BEST LOWER BOUND FOR f^*

NOTE: FROM WEAK DUALITY, WE GET $g^* \leq f^*$

EX. CONT

$$g(\mu) = -\frac{\mu^2}{4} + \mu$$

LAGR. DUAL PROBLEM:

$$g^* = \max_{\text{s.t. } \mu \geq 0} g(\mu)$$

$$= \max_{\text{s.t. } \mu \geq 0} \left[-\frac{\mu^2}{4} + \mu \right]$$

THIS IS SOLVED BY LETTING $\mu^* = 2$, $g^* = 1$

$\Rightarrow$ FOR THIS EX. $g^* = f^*$

THE DUAL PROBLEM

$$\boxed{g^* = \sup g(\mu) \quad \text{s.t. } \mu \geq 0}$$

$$g(\mu) = \min f(x) + \mu^T g(x) \quad \text{s.t. } x \in X$$

THEOREM: THE DUAL FUNCTION g IS CONCAVE AND ITS EFFECTIVE DOMAIN $\{\mu \geq 0 \mid g(\mu) > -\infty\}$ IS CONVEX.

- MAXIMIZING OBJ. FUNC. IN DUAL PROBLEM
- OBJECTIVE FUNCTION TO DUAL PROBLEM IS CONCAVE
- FEASIBLE SET TO — " — IS CONVEX SET

$\Rightarrow$ DUAL PROBLEM IS A CONVEX OPTIMIZATION PROBLEM

PROOF: HOMEWORK

LAGRANGIAN: $L(x, \mu) = f(x) + \mu^T g(x) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x)$

LAGRANGIAN DUAL FUNCTION: $q(\mu) = \min_{x \in X} \left[f(x) + \mu^T g(x) \right] = \min L(x, \mu)$ S.T. $x \in X$

LAGRANGIAN DUAL PROBLEM: $q^* = \sup_{\mu \geq 0} q(\mu)$

DEF: WE CALL μ^* A LAGRANGE MULTIPLIER IF

$$f^* = \inf_{x \in X} L(x, \mu^*) = \inf_{x \in X} \left[f(x) + \mu^{*T} g(x) \right] = \inf_{x \in X} f(x) + \sum_{i=1}^{m} \mu_i^* g_i(x)$$

NOTE: μ^* IS A LAGR. MULT. IF $q(\mu^*) = f^*$

$\rightarrow$ μ^* SOLVES DUAL PROBLEM

$\rightarrow$ ONLY EXIST IF $q^* = f^*$

GLOBAL OPTIMALITY CONDITIONS [NECESSARY & SUFFICIENT]

THM: CONSIDER THE PAIR OF PRIMAL/DUAL VECTORS (x^*, μ^*).
THEN x^* IS OPTIMAL IN PRIMAL PROBLEM AND μ^* IS OPTIMAL IN DUAL PROBLEM (LAGR. MULT.)
IF AND ONLY IF

- $x^* \in \underset{x \in X}{\operatorname{argmin}} \; L(x, \mu^*)$ [LAGRANGIAN OPTIMALITY]
- $\mu^* \geq 0$ [DUAL FEASIBILITY]
- $x^* \in X$, $g_i(x^*) \leq 0$, $i = 1, \dots, m$ [PRIMAL FEASIBILITY]
- $\mu_i^* \cdot g_i(x^*) = 0$, $i = 1, \dots, m$ [COMPLEMENTARY SLACKNESS]

HOMEWORK: LET $f \in C^1$ AND $g_i \in C^1$, $i = 1, \dots, m$ AND TRY TO DIFFERENTIATE $L(x, \mu)$ W.R.T. x
AND SET GRADIENT TO ZERO AS SOLUTION TO a)

STRONG DUALITY

[WEAK DUALITY: $q(\mu) \leq f(x)$
$q^* \leq f^*$]

WHEN DOES $q^* = f^*$ HOLD?? ANSWER: CONVEXITY!

$$f^* = \inf f(x) \quad \text{s.t. } g_i(x) \leq 0, \; i=1,\dots,m, \quad x \in \underline{X}$$

THM: [STRONG DUALITY THEOREM].

ASSUME THAT THERE EXISTS AN INNER POINT TO THE PRIMAL PROBLEM [$\exists x^0 \in \underline{X} : g_i(x^0) < 0, \; i = 1, \dots, m$]. AND THAT $f^* \geqslant -\infty$. ALSO ASSUME THAT f IS A CONVEX FUNCTION, g_i, $i=1,\dots,m$ ARE CONVEX FUNCTIONS AND THAT $\underline{X}$ IS A CONVEX SET. THEN IT HOLDS THAT

$$q^* = f^*$$

PROOF: DIFFICULT!

NOTE: WHEN $q^* = f^*$ WE HAVE NO DUALITY GAP

$$\Delta = f^* - q^*$$