LECTURE 10: LP DUALITY

PRIMAL

$$\boxed{\begin{aligned} z^* = \inf\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}}$$

WHERE $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

- n VARIABLES
- m CONSTRAINTS

DUAL

$$\boxed{\begin{aligned} q^* = \sup\ & b^T y \\ \text{s.t.}\ & A^T y \leq c \\ & y \in \mathbb{R}^m \end{aligned}}$$

- m VARIABLES
- n CONSTRAINTS

LAGRANGIAN DUALITY

$$\boxed{\begin{aligned} z^* = \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}}$$

$$\left[\begin{aligned} \min\ & f(x) \\ \text{s.t. } & \underbrace{g_i(x) \leq 0}_{Ax = b}, \quad i = 1, \dots, m \\ & x \in \underbrace{X}_{x \geq 0} \end{aligned}\right]$$

- LAGRANGIAN RELAX CONSTRAINTS "$Ax = b$"
- INTRODUCE LAGRANGIAN DUAL FUNCTION

$$\begin{aligned} q(y) &= \min_{x \geq 0} \left[c^T x + y^T (b - Ax) \right] \\ &= \min_{x \geq 0} \left[(c - A^T y)^T x + y^T b \right] \\ &= b^T y + \underbrace{\min_{x \geq 0} \left[(c - A^T y)^T x \right]}_{-\infty \text{ UNLESS } c - A^T y \geq 0} \end{aligned}$$

$$= \begin{cases} b^T y & \text{IF } c - A^T y \geq 0 \\ -\infty & \text{OTHERWISE} \end{cases}$$

LAGRANGIAN DUAL PROBLEM:

$$q^* = \sup_{y \text{ FREE}} q(y) = \boxed{\begin{aligned} \sup\ & b^T y \\ \text{s.t. } & A^T y \leq c \\ & y \in \mathbb{R}^m \end{aligned}}$$

DUAL PROBLEMS FOR NON-STANDARD FORM LPs

PRIMAL			DUAL	
OBJECTIVE:	min		OBJECTIVE:	max
VARIABLES:	≥ 0	[CANONICAL]	CONSTRAINTS:	$\leq$
	≤ 0	[NON-CANONICAL]		$\geq$
	FREE			$=$
CONSTRAINTS:	$\geq$	[CANONICAL]	VARIABLES:	≥ 0
	$\leq$			≤ 0
	$=$			FREE

EX.

PRIMAL

$$\min \quad 2x_1 + 3x_2 + x_3$$

$$\begin{aligned}
\text{s.t} \quad x_1 + x_2 + x_3 &\le 1 \quad (y_1) \\
2x_1 \qquad + 4x_3 &= 4 \quad (y_2) \\
x_1 &\ge 0 \\
x_2 &\le 0 \\
x_3 &\text{ FREE}
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 4 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad c = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

DUAL

$$\max \quad y_1 + 4y_2$$

$$\begin{aligned}
y_1 + 2y_2 &\le 2 \qquad [x_1] \\
y_1 \qquad &\ge 3 \qquad [x_2] \\
y_1 + 4y_2 &= 1 \qquad [x_3] \\
y_1 &\le 0 \qquad [\text{NON-CANONICAL}] \\
y_2 &\text{ FREE}
\end{aligned}$$

PRIMAL / DUAL PAIR

$$(P) \quad z^* = \inf \; c^T x \quad \text{s.t.} \; Ax = b, \; x \geq 0$$

$$(D) \quad q^* = \sup \; b^T y \quad \text{s.t.} \; A^T y \leq c, \; y \in \mathbb{R}^m$$

- ALL THEOREMS WILL ASSUME THIS PAIR
- ANALOGOUS RESULTS FOR ALL LP DUAL PAIRS

$$\boxed{\begin{array}{l}\textbf{WEAK DUALITY THEOREM} \\ \text{IF } x \text{ IS A FEASIBLE POINT IN (P)} \\ \text{AND } y \text{ IS A FEASIBLE POINT IN (D) THEN} \\ \qquad c^T x \geq b^T y\end{array}}$$

PROOF:

$$\underline{c^T x} \underset{\substack{\uparrow \\ \left[\begin{array}{c} A^T y \leq c \\ x \geq 0 \end{array}\right]}}{\geq} y^T A x \underset{\substack{\uparrow \\ [Ax = b]}}{=} y^T b = \underline{b^T y}$$

- IT ALSO FOLLOWS THAT

$$z^* \geq q^*$$

COROLLARY:

- IF THE OPTIMAL OBJECTIVE VALUE OF THE PRIMAL PROBLEM (P) IS $-\infty$. THEN THE DUAL PROBLEM (D) IS INFEASIBLE
- IF THE OPT. OBJ. VALUE OF THE DUAL (D) IS $+\infty$ THEN THE PRIMAL PROBLEM (P) IS INFEASIBLE

COROLLARY:

IF x IS FEASIBLE IN (P) AND y IS FEASIBLE IN THE DUAL PROBLEM (D) AND

$$c^T x = b^T y.$$

THEN x IS OPTIMAL IN (P) AND
y IS OPTIMAL IN (D).

- USEFUL FOR OPTIMALITY GUARANTEES.

STRONG DUALITY THEOREM:

IF BOTH (P) AND (D) ARE FEASIBLE, THEN

1. THERE EXISTS x^* OPTIMAL IN (P) AN y^* OPTIMAL IN (D)
2. $c^T x^* = b^T y^*$ (MEANING $z^* = q^*$)

POSSIBILITIES IN P/D

D

P	FINITE OPT	UNBOUNDED	INFEASIBLE
FINITE OPT	X		
UNBOUNDED			X
INFEASIBLE		X	X

CAN HAPPEN

SIMPLEX METHOD

- BFS , $\mathbb{x} = \begin{bmatrix} \mathbb{x}_B \\ \mathbb{x}_N \end{bmatrix}$, $A = [B, N]$, $\mathbb{c} = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$

- FOR A SPECIFIC BFS $(A = [B, N])$

$$\mathbb{x}_B = B^{-1}\mathbb{b}$$

$$\mathbb{x}_N = \mathbb{0}$$

- SIMPLEX METHOD ITERATIVELY UPDATES (B, N) UNTIL OPTIMALITY.
- WHEN OPTIMAL $\quad \zeta_N^T = c_N^T - c_B^T B^{-1} N \geq \mathbb{0}$

- INTRODUCE $\quad \mathbb{y}^* = \left(c_B^T B^{-1}\right)^T$

- $y^* = (c_B^T B^{-1})^T$

- BY DEFINITION $\underbrace{b^T y^*}_{\text{OBJ. VAL IN (D)}} = c_B^T (\underbrace{B^{-1} b}_{x_B}) = c_B^T x_B = c_B^T x_B + c_N^T \underbrace{x_N}_{=0} = \underline{c^T x^*}$

  SO THEY HAVE SAME OBJECTIVE VALUE!

- BUT IS y^* FEASIBLE IN (D)?

$$\left.\begin{array}{ll} c_B^T - (y^*)^T B & = 0 \\ \underbrace{c_N^T - (y^*)^T N}_{\tilde{c}_N^T} & \geq 0 \end{array}\right\} \Rightarrow c^T - (y^*)^T A \geq 0$$

$$\Rightarrow \underline{A^T y^* \leq c}$$

FEASIBLE IN (D)

y^* OPTIMAL IN (D)

REMARKS

- IF YOU SOLVE (P) WITH SIMPLEX METHOD.
  YOU GET AN OPTIMAL SOLUTION TO THE DUAL PROBLEM (D)
  FOR FREE. $y^* = (c_B^T B^{-1})^T$

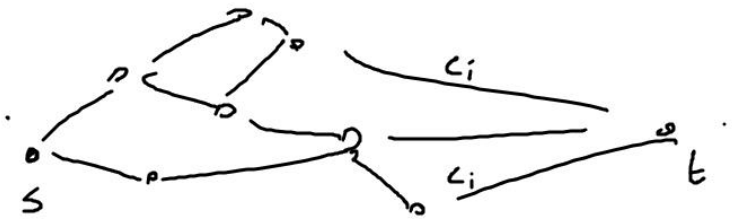
- DIRECTLY PROVIDE OPTIMALITY CERTIFICATE TO YOUR BOSS!

WHAT IS THE DUAL PROBLEM?

EXAMPLE:

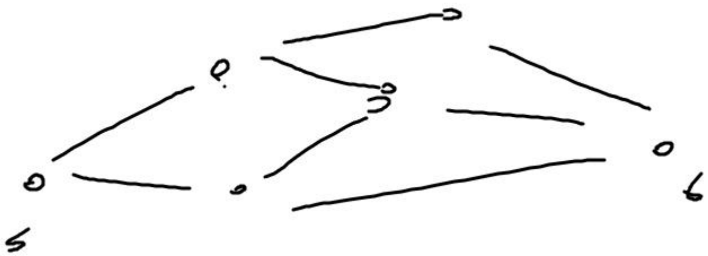
MINIMUM CUT PROBLEM

- A GRAPH WITH EDGE CAPACITIES.
- HOW MUCH CAPACITY DO YOU NEED TO CUT AWAY IN ORDER FOR THERE TO BE NO WAY FROM s TO t



MAXIMUM FLOW PROBLEM

- FIND MAXIMUM FLOW BETWEEN s AND t



COMPLEMENTARITY SLACKNESS THEOREM

LET x BE FEASIBLE IN (P) AN y FEASIBLE IN (D). THEN

$$\left.\begin{array}{l} x \text{ OPTIMAL IN (P)} \\ y \text{ OPTIMAL IN (D)} \end{array}\right\} \iff x_j \left(c_j - A_{\cdot j}^T y \right) = 0 \quad , \; j = 1, \ldots, n$$

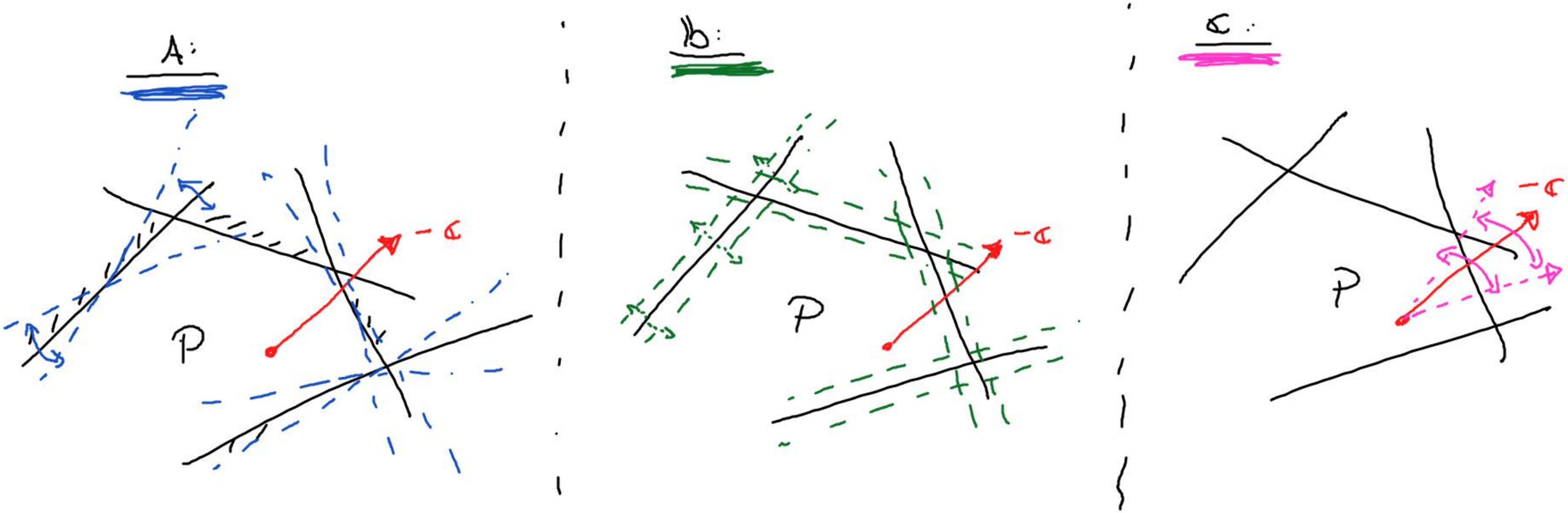
WHERE $A_{\cdot j}$ IS j:TH COLUMN OF A

- IF THERE IS "SLACK" IN THE DUAL CONSTRAINT THEN THE CORRESPONDING PRIMAL VARIABLE MUST BE 0
- IF THE PRIMAL VARIABLE IS POSITIVE, THEN THE CORR. DUAL CONSTRAINT MUST FULFILLED WITH EQUALITY.

PERTURBATIONS

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

A, b, c



PERTURBATIONS IN $\mathbb{b}$

$$v(\mathbb{b}) = \min c^T x \quad \text{s.t. } Ax = \mathbb{b},\ x \geq 0 \qquad (P)$$

$$v(\mathbb{b}') = \min c^T x \quad \text{s.t. } Ax = \mathbb{b}' \left[= \mathbb{b} + \Delta \mathbb{b}\right],\ x \geq 0$$

- CONSIDER PERTURBING $\mathbb{b}$ TO $\mathbb{b} + \Delta\mathbb{b} = \mathbb{b}'$, $v(\mathbb{b}')$?
- SUPPOSE FOR OPTIMAL SOLUTION TO (P), WE HAVE * BFS $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ AND THAT $x_B = B^{-1}\mathbb{b} > 0$ [NON-DEGENERATE]
- THEN IF $\Delta\mathbb{b}$ IS SMALL $x_B' = B^{-1}\mathbb{b}' = B^{-1}\mathbb{b} + B^{-1}\Delta\mathbb{b} = x_B + B^{-1}\Delta\mathbb{b} \geq 0$

  SO STILL FEASIBLE
- AND IT WILL STILL BE OPTIMAL

$$\underline{v(\mathbb{b}')} = c_B^T x_B' = \underbrace{c_B^T B^{-1}}_{y^{*T}}\mathbb{b}' = \underline{(y^*)^T \mathbb{b}'}$$

$\Rightarrow$

<u>SHADOW PRICE THEOREM</u>

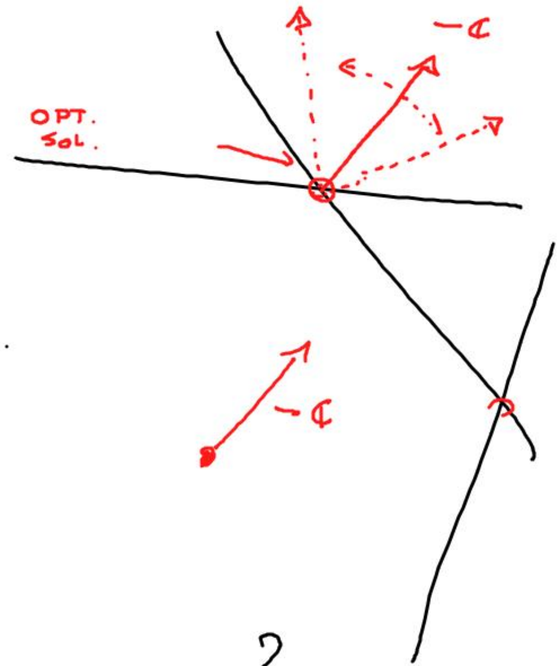
IF, FOR A GIVEN $\mathbb{b}$, THE OPTIMAL BFS OF (P) IS NON-DEGENERATE, THEN

$$\nabla v(\mathbb{b}) = \mathbb{y}^*$$

- IT MEANS THAT CHANGING THE RIGHT HAND SIDE OF (P) WILL AFFECT IN A CHANGE IN OBJECTIVE VALUE WHICH IS $\mathbb{y}^*$.
- SO CHANGING i'TH RIGHT-HAND-SIDE WILL AFFECT THE OPT. OBJECTIVE VALUE WITH $\mathbb{y}_i^*$.

PERTURBATIONS IN $\mathbb{c}$

$$\min \quad (\mathbb{c} + \Delta\mathbb{c})^T \mathbb{x}$$
$$\text{s.t.} \quad A\mathbb{x} = \mathbb{b}$$
$$\mathbb{x} \geq \mathbb{0}$$



- LET $A = [B, N]$ BE OPTIMAL BFS WHE $\Delta C = \mathbb{0}$.
- OPTIMALITY CONDITION

$$\tilde{C}_N^T = C_N^T - C_B^T B^{-1} N \geq 0$$

- WHEN $\Delta \mathbb{c} \neq \mathbb{0}$.

$$\tilde{C}_N^T = (\mathbb{c}_N + \Delta\mathbb{c}_N)^T - (\mathbb{c}_B + \Delta\mathbb{c}_B)^T B^{-1} N \geq \mathbb{0} \quad ?$$

EASY TO CHECK IF OPTIMALITY STILL HOLDS

MORE COMPUTATIONS TO BE DONE