LECTURE 9: SIMPLEX METHOD

RECAP:

- LINEAR PROGRAM: MINIMIZE LINEAR OBJECTIVE FUNCTION OVER A POLYHEDRON
- CONSTANT GRADIENT OF OBJ. FUNC.
  $\Rightarrow$ "PUSHING A PLANE PERPENDICULAR TO $-\nabla f(x)$ AS FAR AS POSSIBLE"

- OPTIMAL SOLUTIONS FOUND IN EXTREME POINTS
- EXTREME POINTS = BASIC FEASIBLE SOLUTION
- ADJACENT ("NEIGHBOURING") BFSs (EXTREME POINTS)
- STANDARD FORM

- STANDARD FORM OF A LP:

$$\boxed{\begin{array}{ll} \min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad , \text{ WHERE } \mathbf{b} \geq \mathbf{0}}$$

WHERE $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$

- ALL LPs CAN BE TRANSFORMED INTO STANDARD FORM.

- A POINT $\bar{\mathbf{x}}$ IS A <u>BASIC FEASIBLE SOLUTION</u> TO THE POLYHEDRON $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ IF

  - $\bar{\mathbf{x}} \geq \mathbf{0}$
  - $A\bar{\mathbf{x}} = \mathbf{b}$
  - THE COLUMNS OF A CORRESPONDING TO NON-ZERO ELEMENTS IN $\bar{\mathbf{x}}$ ARE LIN. INDEP.

- BFS

FOR ANY BFS $\bar{x}$ WE CAN REORDER

$$A = [B, N]$$
$$\bar{x} = [x_B, x_N]$$
$$c = [c_B, c_N]$$

SUCH THAT $\underline{B} \in \mathbb{R}^{m \times m}$, $N \in \mathbb{R}^{m \times (n-m)}$

$\text{rank}(B) = m$

AND

- $\underline{\bar{x}_N = 0}$
- $\underline{\bar{x}_B} = B^{-1} b$

$$\left[\text{SINCE} \quad A\bar{x} = B\bar{x}_B + N \underbrace{x_N}_{0} = b \right]$$

NON-BASIC VARIABLES

BASIC VARIABLES

THE BASIS MATRIX

- <u>BFS</u> (TWO WAYS OF THINKING)

1.

AS A POINT $\bar{x}$ FULFILLING THE BFS REQUIREMENTS

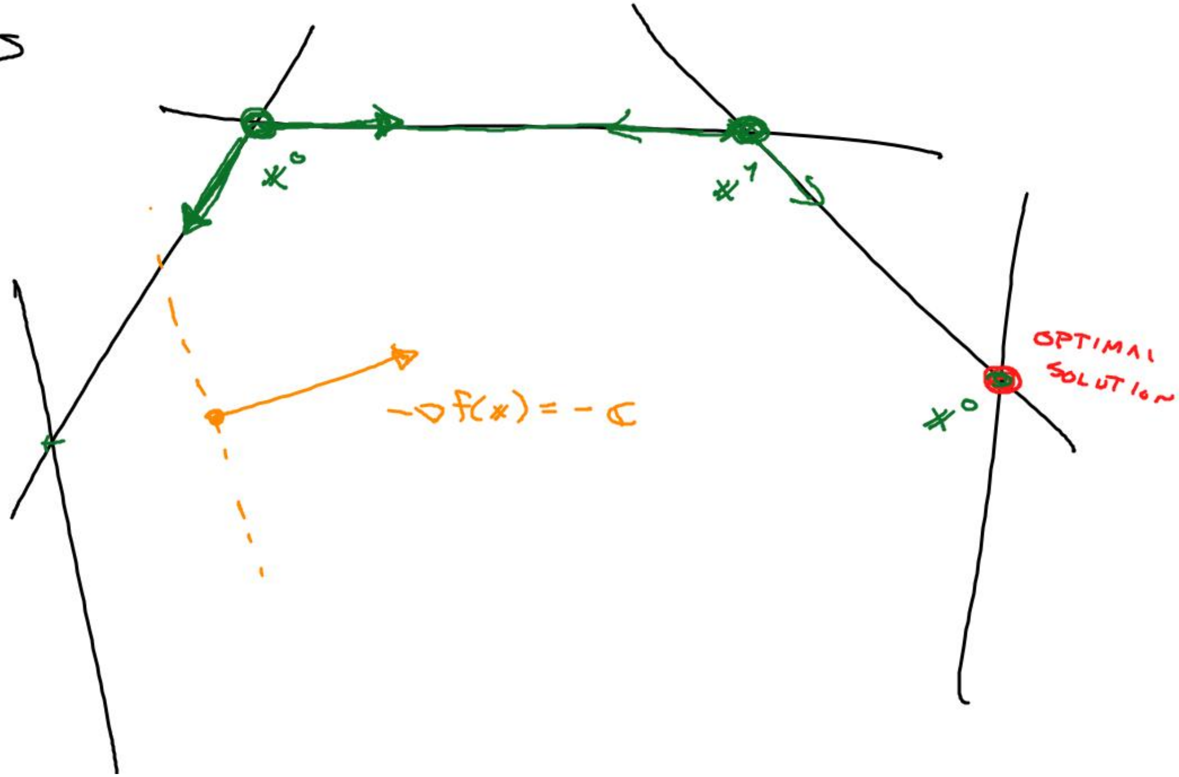
2.

AS A REARRANGEMENT OF $\mathbb{A}$ INTO MATRICES B AND N SUCH THAT $B \in \mathbb{R}^{m \times m}$ AND

$$B^{-1}\mathbb{b} \geq 0$$

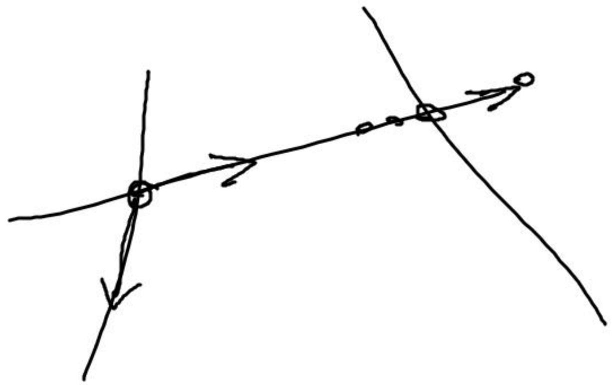
IDEA OF SIMPLEX METHOD

- START IN BFS
- MOVE TO AN ADJACENT BFS
- IF OPTIMAL, STOP
  OTHERWISE, KEEP MOVING



SIMPLEX METHOD

1. DETERMINING IF CURRENT BFS IS OPTIMAL
2. FIND SEARCH DIRECTION (DESCENT)
3. DETERMINE STEP LENGTH
4. UPDATE OUR BFS



ORDER IN LECTURE

2, 3, 4, 1

SEARCH DIRECTION

- ITERATIVE PROCEDURE: $\quad \bar{x} \longrightarrow \bar{x} + \underset{\text{STEP LENGTH}}{\theta}\, \underset{\text{SEARCH DIRECTION}}{d}$

- AT BFS $\bar{x} = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$, WITH $A = [B, N]$, WE CAN ALSO PARTITION THE SEARCH DIRECTION AS $d = \begin{bmatrix} d_B \\ d_N \end{bmatrix}$

- THE IDEA OF SIMPLEX METHOD IS TO MOVE TO ADJACENT EXTREME POINTS, WHICH MEANS $d_N = e_j$ FOR SOME j

UNIT VECTOR $= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

- BUT IF $d_N = e_j$ THEN d_B IS DETERMINED SINCE

$$A(\bar{x} + \theta d) = b \implies A d = 0$$

$[A\bar{x} = b]$

$$\implies \text{SINCE } \boxed{d_N = e_j} \implies \boxed{d_B = -B^{-1} N e_j}$$

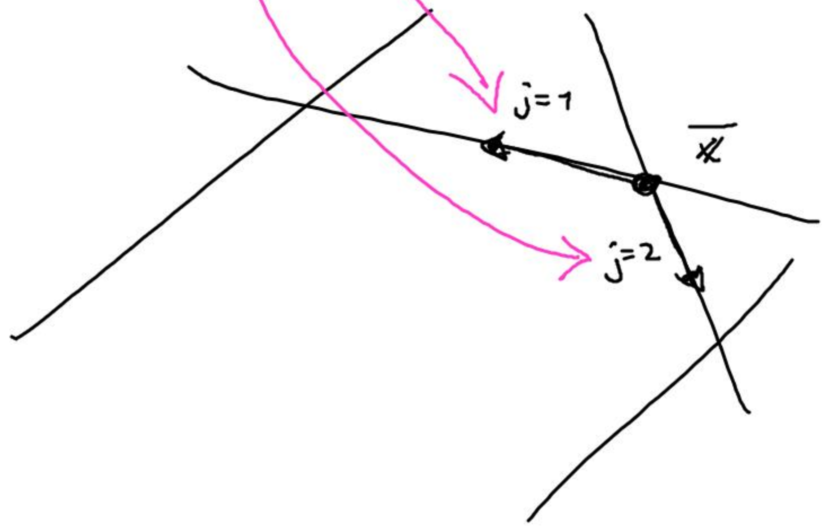
SINCE $[B d_B + N d_N = 0]$

- SO POSSIBLE SEARCH DIRECTIONS ARE

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_B \\ \mathbf{d}_N \end{bmatrix} = \begin{bmatrix} -B^{-1}N\mathbf{e}_j \\ \mathbf{e}_j \end{bmatrix} \quad \text{FOR SOME } j \in \{1, .., n-m\}$$

- REMEMBER THAT $\mathbf{x}_N = \mathbf{0}$

$$\mathbf{x}_B = B^{-1}\mathbf{b}$$



REDUCED COSTS

FROM $\bar{x}$ TO $\bar{x} + \theta d$, THE OBJECTIVE VALUE WILL CHANGE

$$c^T(\bar{x}+\theta d) - c^T\bar{x} = \theta c^T d = \theta \left[c_B^T, c_N^T\right]\begin{bmatrix} d_B \\ d_N \end{bmatrix} = \theta\left[c_B^T, c_N^T\right]\begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix}$$

[CHECK AT HOME]

$$= \theta\left(c_N^T - c_B^T B^{-1} N\right)_j$$

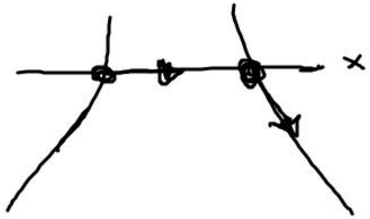
$\Rightarrow$ $\boxed{\tilde{c}_N^T = c_N^T - c_B^T B^{-1} N}$ ~~ARE~~ ARE CALLED THE REDUCED COSTS

$\Rightarrow$ "THEY ARE VALUES OF HOW MUCH THE OBJECTIVE VALUE WILL CHANGE IF WE CHOOSE j-TH NON-BASIC VARIABLE AS "SEARCH DIRECTION" "

- SO TO CHOOSE WHICH SEARCH DIRECTION TO USE WE WILL IN THIS COURSE USE THE <u>MINIMUM REDUCED COST RULE.</u>
- WE CAN CHOOSE $d = \begin{bmatrix} -B^{-1}N e_j \\ e_j \end{bmatrix}$ FOR $j = 1, \dots, n-m$
- WE WILL TAKE THE j WHERE $\underline{\tilde{c}_N^T = c_N^T - c_B^T B^{-1} N}$ IS THE MOST NEGATIVE; i.e.

$$j^* \in \underset{j=1,\dots,n-m}{\operatorname{argmin}} (\tilde{c}^T)_j$$

STEP LENGTH (ASSUME j CHOSEN)



- $\bar{x} = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$
- $\bar{x} + \theta d = \begin{bmatrix} x_B \\ x_N \end{bmatrix} + \theta \begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix} = \begin{bmatrix} B^{-1}b - \theta B^{-1}Ne_j \\ \theta e_j \end{bmatrix} = \begin{bmatrix} B^{-1}b - \theta B^{-1}N_j \\ \theta e_j \end{bmatrix}$
- WHAT CAN GO WRONG? WE CAN MOVE TOO FAR (θ TOO LARGE). WHICH MEANS $\bar{x} + \theta d \not\geq 0$
- IF $B^{-1}N_j \leq 0$, THEN $\bar{x} + \theta d \geq 0$ FOR ALL $\theta \geq 0$. LET $\theta \to \infty$ AND REALIZE THAT THE PROBLEM IS UNBOUNDED
- IF $B^{-1}N_j \not\leq 0$, THEN SOME ENTRY IN $B^{-1}b - \theta B^{-1}N_j$ BECOMES 0 AS θ INCREASES

- CHOOSE θ SUCH THAT ONE ELEMENT IN x_B BECOMES 0 AND THE REST ≥ 0.

- $x_B = B^{-1}b$, $\Longrightarrow$ $x_B + \theta d_B = x_B - \theta B^{-1} N_j$

$$= B^{-1}b - \theta B^{-1} N_j$$

- CHOOSE

$$\boxed{\theta^* = \min_{k : (B^{-1}N_j)_k > 0} \frac{(B^{-1}b)_k}{(B^{-1}N_j)_k}}$$

HOMEWORK

MINIMUM RATIO TEST

LET i^* FULFILLING THE MINIMUM BE THE OUTGOING VARIABLE

CHECKING OPTIMALITY

THM: LET $\bar{x}$ BE A BFS ASSOCIATED WITH BASIS MATRIX B, AND LET $\tilde{c}_N^T = c_N^T - c_B^T B^{-1} N$ BE THE REDUCED COSTS. .

IF $\tilde{c}^T \geq 0 \implies \bar{x}$ IS OPTIMAL SOLUTION

SIMPLEX METHOD

1. ASSUME WE HAVE AN INITIAL BFS. $\bar{x} = [x_B^T, x_N^T]$, $A = [B, N]$

2. COMPUTE REDUCED COSTS $\tilde{c}_N^T = c_N^T - c_B^T B^{-1} N$

   IF $\tilde{c}^T \geq 0$, THEN CURRENT BFS IS OPTIMAL. STOP!

   IF $\tilde{c}^T \ngeq 0$, CHOOSE INDEX $\bar{j}$ THAT HAS ~~LEA~~ MOST NEGATIVE REDUCED COST $(\tilde{c}_N)_j < 0$ AS INCOMMING VARIABLE

3. COMPUTE $B^{-1} N_j$.

   IF $B^{-1} N_j \leq 0$, THEN PROBLEM IS UNBOUNDED, STOP!

   IF $B^{-1} N_j \nleq 0$, COMPUTE $i \in \underset{k:(B^{-1}N_j)_k > 0}{\text{argmin}} \frac{(B^{-1}b)_k}{(B^{-1}N_{\bar{j}})_k}$ AND LET $\bar{i}$ BE OUTGOING VARIABLE

4. UPDATE B AND N ~~BY~~ BY LIFTING IN COLUMN $\bar{j}$ FROM N INTO B AND LETTING COLUMN $\bar{i}$ MOVE FROM B TO N.

   GO TO STEP 2.

THM: IF P IS NONEMPTY AND EVERY BFS IS NON-DEGENERATE THEN THE SIMPLEX METHOD TERMINATE AFTER FINITELY MANY STEPS. AT TERMINATION, TWO POSSIBILITIES:

a) AN OPTIMAL BASIS B IS FOUND

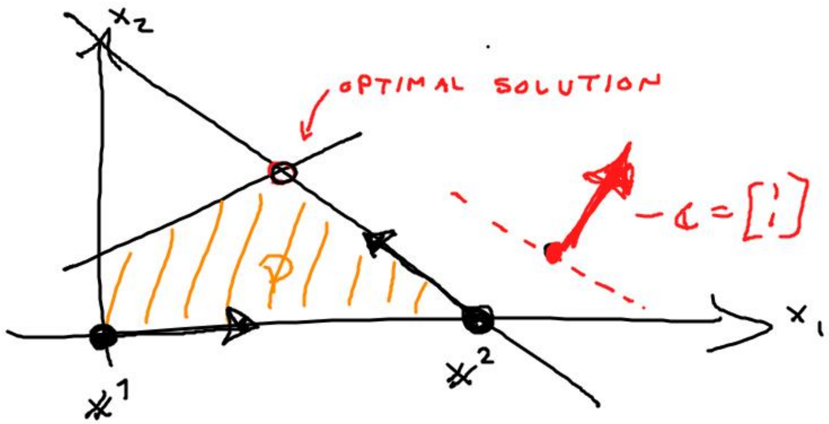
b) THE PROBLEM WAS FOUND TO BE UNBOUNDED

NOTES:

- WORKS VERY GOOD IN PRACTICE
- IT CAN, IN WORST CASE, BE EXPONENTIAL
- $\binom{n}{m}$ ARE UPPER BOUND OF # BFSs
- POLYNOMIAL TIME ALG. EXISTS FOR LPs

EX:

$$\min \quad -x_1 - x_2$$
$$\text{s.t.} \quad 2x_1 + x_2 \le 2$$
$$-x_1 + x_2 \le 1/2$$
$$x_1, x_2 \ge 0$$



STANDARD FORM

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 + s_1 = 2 \\ & -x_1 + x_2 + s_2 = 1/2 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

INITIAL BFS: $\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1/2 \end{bmatrix}$ $\quad \mathbf{x}_B = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ $\quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

ITERATION 1:

$$x_B = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_B = B^{-1} b = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} \geq 0 \quad \text{OK!}$$

$$x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad c_N = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad x_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

INCOMING VARIABLE:

$$\tilde{c}^T = c^T - c_B^T B^{-1} N = [-1, -1] - [0, 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = [-1, -1]$$

CHOOSE x_1 AS INCOMING

OUTGOING VARIABLE:

$$B^{-1} N_{x_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \not\leq 0, \quad \text{NOT UNBOUNDED}$$

$$\min_{k : (B^{-1} N_j)_k > 0} \frac{(B^{-1} b)_k}{(B^{-1} N_j)_k} = \frac{2}{2}$$

$$B^{-1} b = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$

$$B^{-1} N_j = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

CHOOSE s_1 AS OUTGOING

UPDATE BASIS: s_1 OUT FROM BASIS, x_1 IN TO BASIS

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbb{b} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}, \quad \mathbb{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

ITERATION 2:

$$\mathbb{x}_B = \begin{bmatrix} x_1 \\ s_2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad \mathbb{c}_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \mathbb{x}_B = B^{-1}\mathbb{b} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \geq \mathbb{0}$$

OK!

$$\mathbb{x}_N = \begin{bmatrix} x_2 \\ s_1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbb{c}_N = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \mathbb{x}_N = \mathbb{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

INCOMING:

$$\tilde{\mathbb{c}}_N^T = \mathbb{c}_N^T - \mathbb{c}_B^T B^{-1} N = \begin{bmatrix} -1/2, & 1/2 \end{bmatrix}$$

CHOOSE x_2 AS INCOMING

OUTGOING:

$$B^{-1}N_{x_2} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \not\leq \mathbb{0}$$

, so not unbounded

MIN RATIO TEST $\quad \min\limits_{k:(B^{-1}N_j)_k > 0} \dfrac{(B^{-1}\mathbb{b})_k}{(B^{-1}N_{x_2})_k} : \dfrac{1}{1/2}, \dfrac{3/2}{3/2}$

CHOOSE S_2 AS OUTGOING

UPDATE BASIS: $\quad s_2$ OUT FROM BASIS, x_2 IN

ITERATION 3:

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad c_B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_B = B^{-1}\mathbf{b} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \geq \mathbf{0} \quad \text{ok!}$$

$$\mathbf{x}_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{c}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

INCOMING: $\tilde{c}_N^T = c_N^T - c_B^T B^{-1} N = \left[2/3, \ 1/3\right] \geq \mathbf{0}$, OPTIMAL BASIS! STOP!

TRANSFORM THIS INTO OUR ORIGINAL PROBLEM

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1}\mathbf{b} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ $\boxed{\mathbf{x}^* = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \text{ IS OPTIMAL IN ORIGINAL PROBLEM}}$