LECTURE 8: LINEAR PROGRAMMING (LP) - INTRO & GEOMETRY

$$\min f(x) \quad \text{s.t. } x \in S \implies \left[\begin{array}{l} f \text{ IS LINEAR} \\ S \text{ IS A POLYHEDRON} \end{array}\right] \implies \boxed{\begin{array}{l} \min\ c^T x \\ \text{s.t. } x \in P \end{array}}$$

LINEAR PROGRAM

WHERE P IS A POLYHEDRON

REMINDER: A SET $P \subseteq \mathbb{R}^n$ IS A POLYHEDRON IF IT CAN BE WRITTEN AS

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

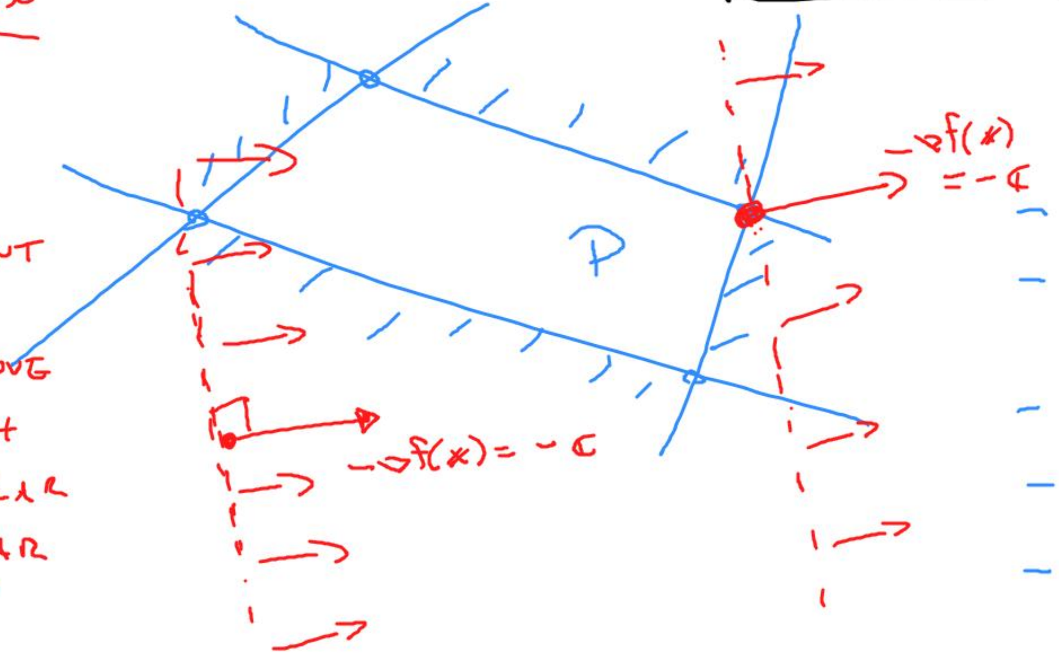
FOR SOME A AND b.

ROADMAP / IDEAS

$$\min\ c^T x \quad \text{s.t.}\ x \in P$$

OBJECTIVE FUNCTION

- $f(x) = c^T x$
- $\nabla f(x) = c$
- CONSTANT GRADIENT
- "TRY TO TO MOVE A PLANE WHICH IS PERPENDICULAR TO $-c$ AS FAR AS POSSIBLE"
- "OPTIMAL SOLUTIONS SHOULD EXIST IN EXTREME POINTS"



SET:

- POLYHEDRON
- INTERSECTION OF HALF-SPACES
- CAN BE UNBOUNDED
- "LINEAR BOUNARIES"
- "DISTINCT CORNER POINTS"

SOLUTION APPROACHES

- MOVE PLANE AS FAR AS POSSIBLE

INTERIOR POINT METHOD

- SEARCH IN EXTREME POINTS

SIMPLEX METHOD !

- WE WILL USE POLYHEDRA ON <u>STANDARD FORM</u>

$$P = \{ x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0 \}$$

- AN LP ON <u>STANDARD FORM</u> IS

$$\boxed{\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}} \qquad \text{WHERE} \quad b \geq 0$$

- THIS CAN MODEL <u>ALL LPs</u>

EX: STD FORM

$$\boxed{\begin{aligned} \min\ & \mathbf{c}^T\mathbf{x} \\ \text{s.t. } & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}}$$

$$\begin{aligned} \min\ & -x_1 + 4x_2 \\ \text{s.t. } & -2x_1 + x_2 \leq 1 \\ & x_1 - x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\Longrightarrow$ INTRODUCE "SLACK VARIABLES". $\Rightarrow$

$$\begin{aligned} \min\ & -x_1 + 4x_2 \\ \text{s.t. } & -2x_1 + x_2 + s_1 = 1 \\ & x_1 - x_2 - s_2 = 1 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

$$\mathbf{c} = [-1, 4, 0, 0]^T, \quad \mathbf{x} = [x_1, x_2, s_1, s_2]^T$$

$$\Longrightarrow \quad A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = [1, 1]^T$$

EX.

$$(P)\quad \begin{aligned} \min\ & \mathbb{c}^T \mathbb{x} \\ \text{s.t.}\ & A\mathbb{x} \leq \mathbb{b} \\ & \mathbb{x} \geq \mathbb{0} \end{aligned} \qquad \longrightarrow \qquad (P')\quad \begin{aligned} \min\ & \mathbb{c}^T \mathbb{x} \\ \text{s.t.}\ & A\mathbb{x} + \mathbb{s} = \mathbb{b} \\ & \mathbb{x}, \mathbb{s} \geq \mathbb{0} \end{aligned}$$

- $\mathbb{x}^*$ OPTIMAL TO (P) $\Longleftrightarrow$ $\exists\, \mathbb{s}^*$ SUCH THAT $[\mathbb{x}^*, \mathbb{s}^*]$ OPTIMAL IN P'

- OTHER "TRICKS" FOR TRANSFORMING LPs TO LPs ON STANDARD FORM.
  YOU WILL LEARN THEM BY DOING EXERCISES!

- ASSUME THAT $P = \{ x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0 \}$, $b \geq 0$
- ASSUME THAT $\operatorname{rank}(A) = m$, $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$

DEF: A POINT $\bar{x}$ IS A **BASIC SOLUTION** IF

- $A\bar{x} = b$
- THE COLUMNS OF A CORRESPONDING TO NON-ZERO ELEMENTS OF $\bar{x}$ ARE LINEARLY INDEPENDENT

PROCEDURE FOR FINDING BASIC SOLUTIONS:

1. CHOOSE m LINEARLY INDEPENDENT COLUMNS OF A
2. REARRANGE $A = [B, N]$ WHERE $B \in \mathbb{R}^{m \times m}$ AND $\text{rank}(B) = m$

   WHERE B CONSISTS OF THE CHOSEN COLUMNS.

   LET N BE THE REST OF THE COLUMNS.

$$A\mathbf{x} = \underbrace{B\mathbf{x}_B + N\mathbf{x}_N}_{[A\ N]\begin{bmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{bmatrix}} = \mathbf{b}$$

WHERE $\mathbf{x} = [\mathbf{x}_B, \mathbf{x}_N]$ ACCORDING TO CHOSEN COLUMNS

3. SET $\mathbf{x}_N = \mathbf{0}$ [NON-BASIC VARIABLES]
4. SET $\mathbf{x}_B = B^{-1}\mathbf{b}$ [BASIC VARIABLES]

HOMEWORK:

SHOW THAT $\mathbf{x} = [\mathbf{x}_B, \mathbf{x}_N]$ IS A BASIC SOLUTION

DEF: A POINT $\bar{x}$ IS A BASIC FEASIBLE SOLUTION (BFS) IF

- $\bar{x} \geq 0$
- $A\bar{x} = b$
- THE COLUMNS OF A CORR. TO NON-ZERO ELEMENTS IN $\bar{x}$ ARE LINEARLY INDEPENDENT

- A BFS IS JUST A BASIC SOLUTION WHICH IS NON-NEGATIVE
- REMEMBER: $P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \}$.
- A BFS IS A POINT IN P.

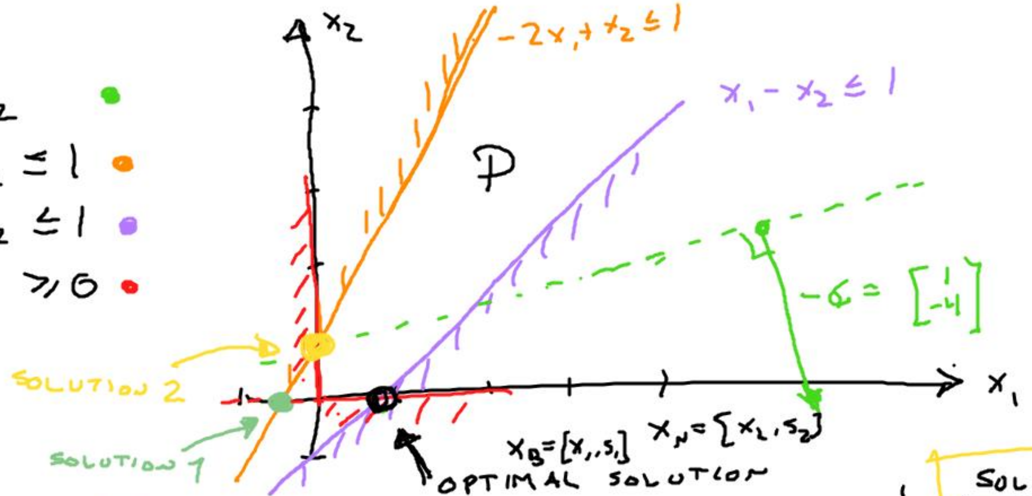
THM: ASSUME $\text{rank}(A) = m$. A POINT $\bar{x}$ IS AN **EXTREME POINT** TO THE POLYHEDRON $P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ IF AND ONLY IF IT IS A **BASIC FEASIBLE SOLUTION**

"BFS" = "EXTREME POINT"

- CAN NOW CHECK EASILY IF CANDIDATE POINT IS EXTREME POINT TO OUR POLYHEDRON $\Rightarrow$ JUST CHECK IF ITS A BFS
- WE HAVE A ~~WAY~~ PROCEDURE OF PRODUCING BASIC SOLUTIONS. IF WE GET LUCKY OUR BASIC SOLUTION MIGHT BE A BFS

EX.

$$\min \; -x_1 + 4x_2$$
$$\text{s.t.} \quad -2x_1 + x_2 \leq 1$$
$$x_1 - x_2 \leq 1$$
$$x_1, x_2 \geq 0$$



REWRITE ON STD FORM

$$\begin{cases} \min \; -x_1 + 4x_2 \\ \text{s.t.} \quad -2x_1 + x_2 + s_1 = 1 \\ \qquad x_1 - x_2 + s_2 = 1 \\ \qquad x_1, x_2, s_1, s_2 \geq 0 \end{cases}$$

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix}$$

SOLUTION 1

SELECT COLS 1 & 4

$$B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$x_B = [x_1, s_2], \quad x_N = [x_2, s_1] \quad \Rightarrow \quad x = \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 3/2 \end{bmatrix} \not\geq 0$$

$$x_N = [0 \;\; 0]$$

$$x_B = B^{-1} b = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$$

NOT BFS

SOLUTION 2

SELECT COLS 2 & 4

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$x_B = [x_2, s_2], \quad x_N = [x_1, s_1] \quad \Rightarrow \quad x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \geq 0$$

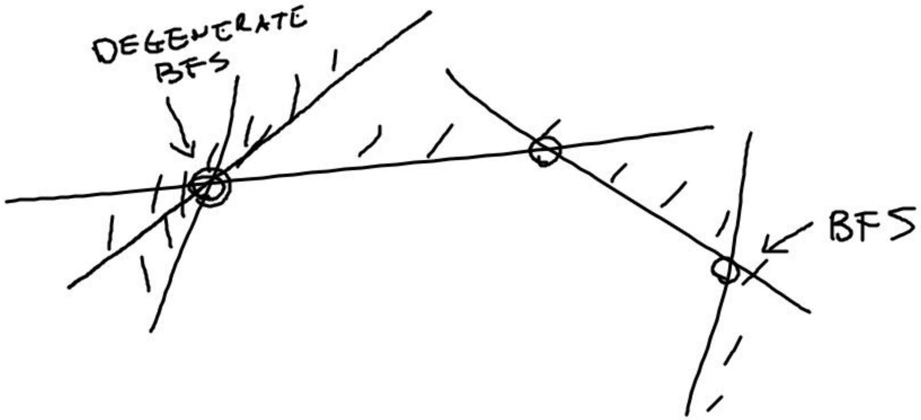
$$x_N = [0 \;\; 0]$$

$$x_B = B^{-1} b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

BFS!

DEGENERATE BFSs

- CONSIDER A BFS $\bar{x} = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ WITH BASIS B $\left[A = [B, N] \right]$
- BY DEFINITION $x_N = 0$ AND $x_B = B^{-1} b$
- BFS $\bar{x}$ IS DEGENERATE IF SOME ELEMENTS IN x_B ARE ZERO.
- IDEA:

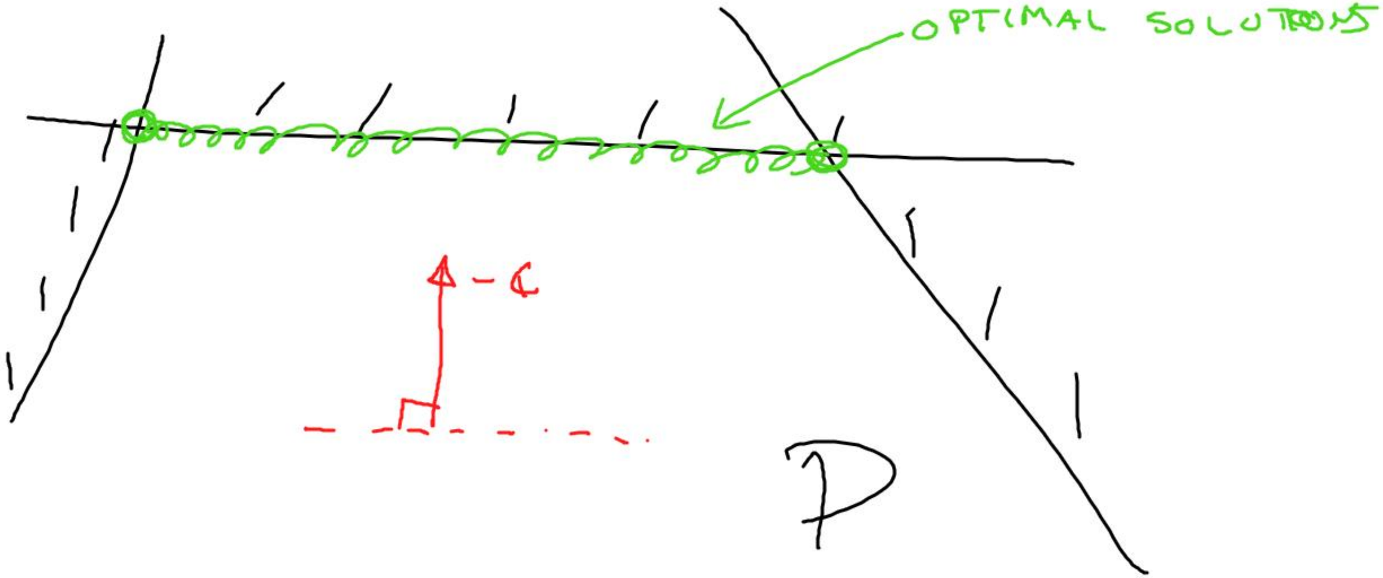


THM: CONSIDER $z^* = \inf z = c^T x$ s.t. $x \in P$ WHERE P IS A POLYHEDRON.. $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

a) THIS PROBLEM HAS A FINITE OPTIMAL SOLUTION IF AND ONLY IF P IS NONEMPTY AN z IS BOUNDED ON P, MEANING THAT $c^T d_j \geq 0$ FOR ALL $d_j \in D$ (THE SET OF EXTREME DIRECTIONS OF $\{x \in \mathbb{R}^n \mid Ax = 0, x \geq 0\}$).

b) MOREOVER, IF THE PROBLEM HAS A FINITE OPTIMAL SOLUTION, THEN THERE EXISTS AN OPTIMAL SOLUTION AMONG THE EXTREME POINTS OF P.

NOT ONLY EXTREME POINTS CAN BE OPTIMAL.

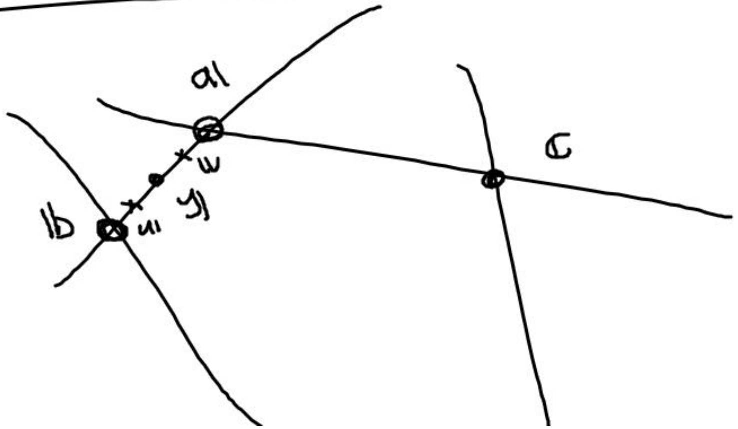


DEF: TWO BFSs $\mathbb{a}$ AND $\mathbb{b}$ ARE ADJACENT IF

$$\forall \mathbb{y} \in \alpha \mathbb{a} + (1-\alpha)\mathbb{b}, \quad \alpha \in (0,1):$$

$$\mathbb{y} = \lambda \mathbb{u} + (1-\lambda)\mathbb{v}, \quad \mathbb{u}, \mathbb{v} \in P, \quad \lambda \in (0,1)$$

$$\Longrightarrow \begin{cases} \mathbb{u} = \alpha_u \mathbb{a} + (1-\alpha_u)\mathbb{b}, & \alpha_u \in (0,1) \\ \mathbb{v} = \alpha_v \mathbb{a} + (1-\alpha_v)\mathbb{b}, & \alpha_v \in (0,1) \end{cases}$$



$\mathbb{a}$ & $\mathbb{b}$ ADJACENT

$\mathbb{a}$ & $\mathbb{c}$ ADJACENT

$\mathbb{b}$ & $\mathbb{c}$ NOT ADJACENT

THM: LET u & w BE BFSs. CORRESPONDING TO (B^1, N^1), (B^2, N^2), RESPECTIVELY. ASSUME THAT ALL BUT ONE COLUMN IN B^1 AND B^2 ARE THE SAME. THEN u AND w ARE ADJACENT BFSs

- ONLY ONE COLUMN SHOULD DIFFER BETWEEN TWO BASIC MATRICES B^1 AND B^2 IN ORDER FOR THE BFSs TO BE ADJACENT.

NEXT LECTURE

- SIMPLE DESCRIPTION OF EXTREME POINTS (BFS)
- "PROCEDURE" FOR FINDING THEM
- DEFINITION OF ADJACENCY AND A WAY OF CHECKING THIS
- KNOW THAT WE SHOULD SEARCH IN EXTREME POINTS

$\Rightarrow$ SIMPLEX METHOD!