Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	10–04–06
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0703-088304)
Result announced:	10–04–23 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{minimize} \quad & z = -2x_1 + x_2 \\
\text{subject to} \quad & x_1 - 3x_2 \leq \beta, \\
& 0 \leq x_1, \\
& 0 \leq x_2 \leq 2.
\end{aligned}
$$

(2p) a) Solve this problem for $\beta = -3$ by using phase I and phase II of the simplex method.

[*Aid:* Utilize the identity

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
$$

for producing basis inverses.]

(1p) b) Without solving any additional linear programs, state, with a clear motivation, the marginal change in the optimal objective function value when β is varied from its current value of -3 (i.e., state the partial derivative of z^* with respect to β).

Question 2

(implications in theorems)

The following questions consider optimality conditions and theorems related to them. Your task is to find counter-examples showing that some theorems formulated in terms of a given implication is only valid in that direction, and not in the reverse direction.

(1p) a) An elementary result for linear programs (LPs) says that if the program has a finite optimal solution, then there exists an optimal solution among the extreme points. Show that it is not necessary for an optimal solution to an LP to be an extreme point by constructing a specific LP counter-example.[1]

(1p) b) An optimality condition for twice differentiable functions is that if $\nabla f(\boldsymbol{x}^*) =$

[1]The problem formulation in the exam was unclear, and has been modified here.

$\mathbf{0}$ and also $\nabla^2 f(\boldsymbol{x}^*)$ is positive definite, then $\boldsymbol{x}^*$ is a strict local minimum of f on $\mathbb{R}^n$. By presenting a counter-example, show that the reverse implication does not hold true in general

(1p) c) Show, by presenting a specific problem of the form $\min_{x \in S} f(\boldsymbol{x})$, where f is differentiable and S is convex, that the variational inequality is not a sufficient condition for local optimality (it is only a necessary condition). (Recall that the variational inequality considers scalar products of the gradient of f at the point of interest and vectors from the point into the set.)

(3p) Question 3

(modelling)

A cylindrical heat storage unit of diameter D and height H is to be constructed. The heat loss due to convection is $h_c = k_c A(T - T_O)$ and due to radiation is $h_r = k_r A(T^4 - T_O^4)$, where k_c and k_r are constants, A is the surface area of the heat storage, T is the temperature inside the heat storage and T_O is the outside temperature. The heat energy stored in the unit is given by $Q = kV(T - T_O)$, where k is a constant and V is the volume of the heat storage.

Formulate an optimization problem for finding the dimensions of a heat storage such that the heat loss is minimized, at least a given constant Q' of heat is stored, and the storage fits inside a sphere of radius R. Your variables, constants, constraints and objective function should be clearly defined.

Is your model best described as a linear programming, nonlinear programming or mixed integer programming model?

Question 4

(linear programming duality)

Consider the linear programming problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c}$, $\boldsymbol{l}$, $\boldsymbol{u}$, and $\boldsymbol{x}$ are vectors in $\mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$.

(1p) a) Give the LP dual of this problem.

(1p) b) Prove that this LP dual always has feasible solutions.

(1p) c) What can you conclude if the primal problem has feasible solutions?

Question 5

(Newton's algorithm)

An engineer has decided to verify numerically that the exponential function $x \mapsto \exp(x) = e^x$ grows faster than any polynomial. In order to do so he/she studies the optimization problem to

$$\text{minimize } f(x) = x^\alpha - exp(x), \tag{1}$$

where α is the highest power of the polynomial (we assume it is an even, positive integer number). The engineer uses a Newton method (with unit steps!) to solve the problem. He/she argues that if the exponential function grows faster than any polynomial, then the sequence $\{x_k\}$ generated by the method should converge to infinity, because the objective function f can be decreased indefinitely by increasing the value of x.

(1p) a) State the Newton iteration explicitly for the given problem (1).

(1p) b) Construct a numerical example (that is, choose a value of $\alpha \in \{2, 4, \ldots\}$ and a starting point of the Newton algorithm) illustrating the engineer's error in reasoning.

(1p) c) Find the error in the engineer's reasoning and formally explain it.

(3p) Question 6

(fundamental theorem of global optimality)

Consider the problem to

$$\text{minimize } f(\boldsymbol{x}), \tag{1a}$$
$$\text{subject to } \boldsymbol{x} \in S, \tag{1b}$$

where $S \subseteq \mathbb{R}^n$ is a nonempty set and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a given function.

Establish the validity of the following theorem.

Consider the problem (1)*, where S is a convex set and f is convex on S. Then, every local minimum of f over S is also a global minimum.*

Question 7

(optimality conditions)

Consider the problem to project (according to the standard Euclidean distance) the vector $\boldsymbol{z} = (2, 3/2)^{\mathrm{T}}$ onto the set S specified by the constraints that $x_j \geq 0$ for $j = 1, 2$, and that $x_1 + x_2 \leq 3/2$.

(1p) a) Describe the appropriate optimization problem to be solved in order to find this projection, and establish that it is a convex problem with a strictly convex objective function.

(1p) b) State the KKT conditions corresponding to a feasible vector $\boldsymbol{x}^*$ being stationary in the problem in a). Establish whether or not the KKT conditions are necessary for a local minimum at $\boldsymbol{x}^*$, and also whether the KKT conditions are sufficient for a feasible vector $\boldsymbol{x}^*$ satisfying the KKT conditions to be a global minimum of the same problem.

(1p) c) Establish whether or not the vector $\boldsymbol{x} = (1, 1/2)^{\mathrm{T}}$ is the projection of $\boldsymbol{z}$ onto the set S.

Good luck!