LECTURE 14: CONSTRAINED OPTIMIZATION

$$\boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}} \quad \xleftrightarrow{\text{TRANSFORM}} \quad \boxed{\begin{array}{ll} \min & f(x) + \chi_S(x) \\ \text{s.t.} & x \in \mathbb{R}^n \end{array}}$$

WHERE

$$\chi_S(x) = \begin{cases} 0 & \text{IF } x \in S \\ +\infty & \text{OTHERWISE} \end{cases}$$

INDICATOR FUNCTION OF THE SET S

IDEA OF PENALTY METHODS

PENALTY METHODS

$$\boxed{\begin{array}{ll} \min & f(x) + \chi_S(x) \\ \text{s.t.} & x \in \mathbb{R}^n \end{array}}$$

- FEASIBILITY IS TOP PRIORITY!

  ONLY WHEN WE ARE FEASIBLE CAN START CONSIDERING f.

- COMPUTATIONALLY HORRIBLE!

  NON-DIFFERENTIABLE, DISCONTINUOUS, NOT FINITE

- DO NOT GET INFORMATION REGARDING HOW INFEASIBLE A POINT IS!

- IDEA: APPROXIMATE THE INDICATOR FUNCTION!

EXTERIOR PENALTY METHOD

- SUPPOSE

$$\left.\begin{aligned} &\min \ f(x) \\ &\text{s.t. } \ g_i(x) \leq 0 \ , \ i=1,\dots,m \\ &\qquad h_j(x) = 0 \ , \ j=1,\dots,\ell \end{aligned}\right\} S$$

- CHOOSE A PENALTY FUNCTION $\psi : \mathbb{R} \to \mathbb{R}_+ : \quad \psi(s) = 0 \iff s = 0$

  EX: $\psi_1(s) = |s| \ , \quad \psi_2(s) = s^2$

- APPROXIMATE INDICATOR FUNCTION $\chi_S(x)$:

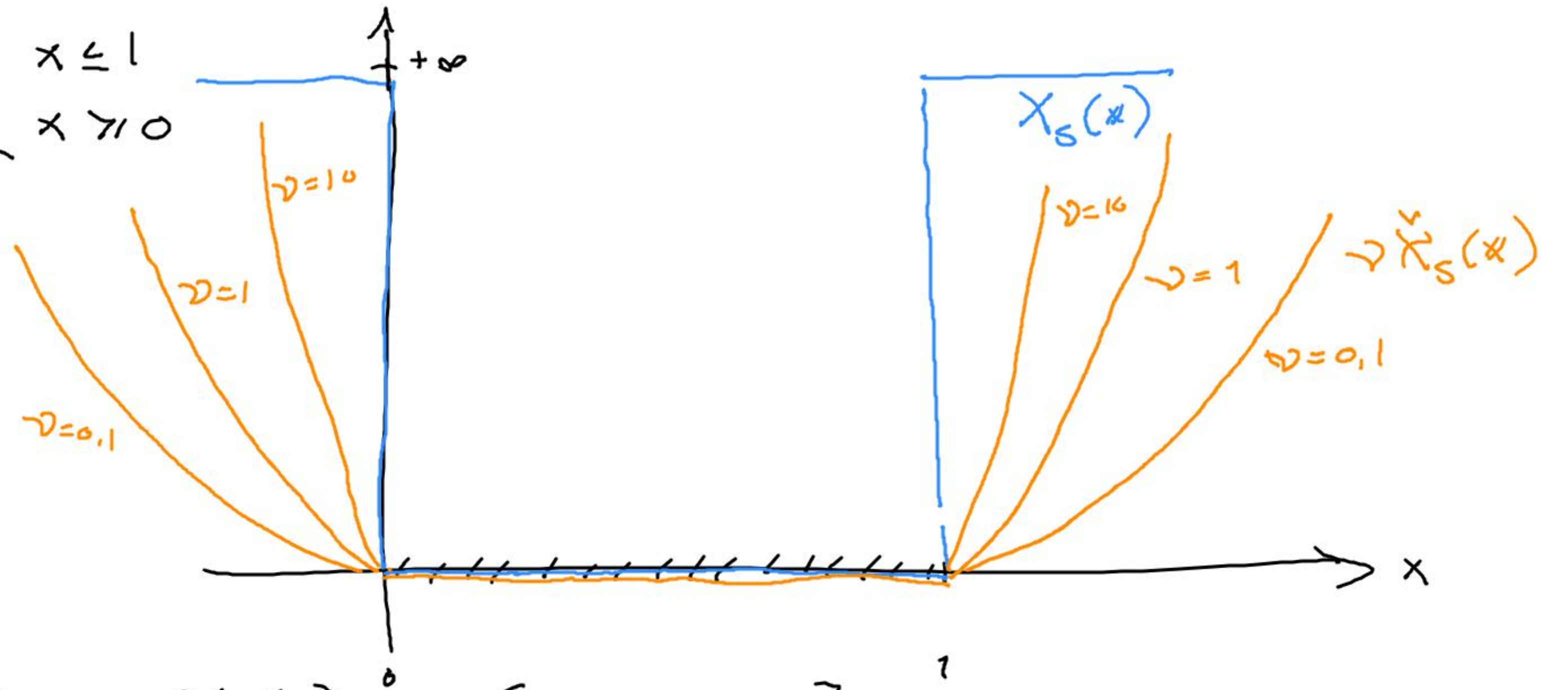
$$\chi_S(x) \approx \nu \check{\chi}(x) = \nu \left[\sum_{i=1}^{m} \psi(\max(0, g_i(x))) + \sum_{j=1}^{\ell} \psi(h_j(x)) \right]$$

ν : PENALTY PARAMETER "≥ 0"

IF x FESIBLE $\Rightarrow \nu \check{\chi}_S(x) = 0$

OTHERWISE $\Rightarrow \nu \check{\chi}_S(x) > 0$

EX: $S = \begin{cases} x \le 1 \\ x \geq 0 \end{cases}$



NOTE:

- $\nu \check{\chi}_S(x) \le \chi_S(x)$ [LOWER BOUND]
- $\nu \to \infty$, THEN $\nu \check{\chi}_S(x) \to \chi_S(x)$
- GIVES INFORMATION REGARDING HOW INFEASIBLE A POINT IS

CONVERGENCE

- CONSIDER A SEQUENCE $\{\nu_k\}$ WITH $\lim_{k \to \infty} \nu_k = +\infty$
- CORRESPONDING TO SUBPROBLEMS

$$\min_{x \in \mathbb{R}^n} f(x) + \nu_k \check{\chi}_S(x)$$

WITH OPTIMAL SOLUTION $x^*_{\nu_k}$

- IF $\{x^*_{\nu_k}\}$ HAS A LIMIT POINT $\hat{x}$, THEN $\hat{x}$ IS AN OPTIMAL SOLUTION TO THE PROBLEM

$$\min f(x) \quad \text{s.t. } x \in S$$

- BASICALLY ONLY USEFUL IF f AND S ARE CONVEX!

MORE USEFUL CONVERGENCE RESULTS

- LET $f, g_i, h_j \in C^1$

$$(P) \quad \min f(x) \quad \text{s.t.} \quad g_i(x) \leq 0, \; i=1,\dots,m \qquad h_j(x) = 0, \; j=1,\dots,\ell$$

THEOREM: ASSUME THAT THE PENALTY FUNCTION ψ IS IN C^1 AND THAT $\psi'(s) \geq 0$ FOR ALL $s \geq 0$. CONSIDER A SEQUENCE $\nu_k \to \infty$

- x_k STATIONARY POINT IN $\min_{x \in \mathbb{R}^n} f(x) + \nu_k \check{\chi}_S(x)$
- $x_k \to \hat{x}$ AS $k \to \infty$
- LICQ HOLDS AT $\hat{x}$
- $\hat{x}$ IS FEASIBLE ($\hat{x} \in S$)

$\Rightarrow$ $\hat{x}$ IS A KKT POINT IN THE PROBLEM (P)

COMPUTATIONAL CONSIDERATIONS

- ν LARGE $\Rightarrow$ $f(x) + \nu \check{\chi}_S(x)$ IS NUMERICALLY DIFFICULT

- IDEA IS TO START WITH SMALL ν AND INCREASE IT SLOWLY.

<u>WHY?</u>

- IF $\bar{x}_{\nu_k}$ IS A STATIONARY POINT WHEN $\nu = \nu_k$ THEN IF $\nu_{k+1} \approx \nu_k$ WE HOPE THAT $\bar{x}_{\nu_{k+1}} \approx \bar{x}_{\nu_k}$

- BUT THEN WE CAN <u>START IN</u> $\bar{x}_{\nu_k}$ WHEN WE SOLVE THE SUBPROBLEM WITH ν_{k+1}

RECAP: EXTERIOR PENALTY METHOD

- TRANSFORM CONSTRAINED OPT. PROBLEM

$$\min f(x) \quad \text{s.t. } g_i(x) \le 0 \quad i=1,\dots,k, \qquad h_j(x)=0 \quad j=1,\dots,l$$

  TO A SEQUENCE OF UNCONSTRAINED PROBLEMS $\min_{x \in \mathbb{R}^n} f(x) + \nu_k \check{\chi}_S(x)$

- APPROXIMATING INDICATOR FUNCTION FROM BELOW
- AS $\nu_k \to \infty$, $x^*_{\nu_k} \to x^*$
- AS $\nu_k \to \infty$, $\underset{\text{STATIONARY}}{\bar{x}_{\nu_k}} \to \underset{\text{KKT}}{\bar{x}}$
- INCREASE ν_k SMOOTHLY.

INTERIOR PENALTY METHOD

- CONSIDER $\boxed{\begin{array}{l} \min f(x) \\ \text{s.t. } g_i(x) \leq 0,\ i=1,\dots,m \end{array}}$

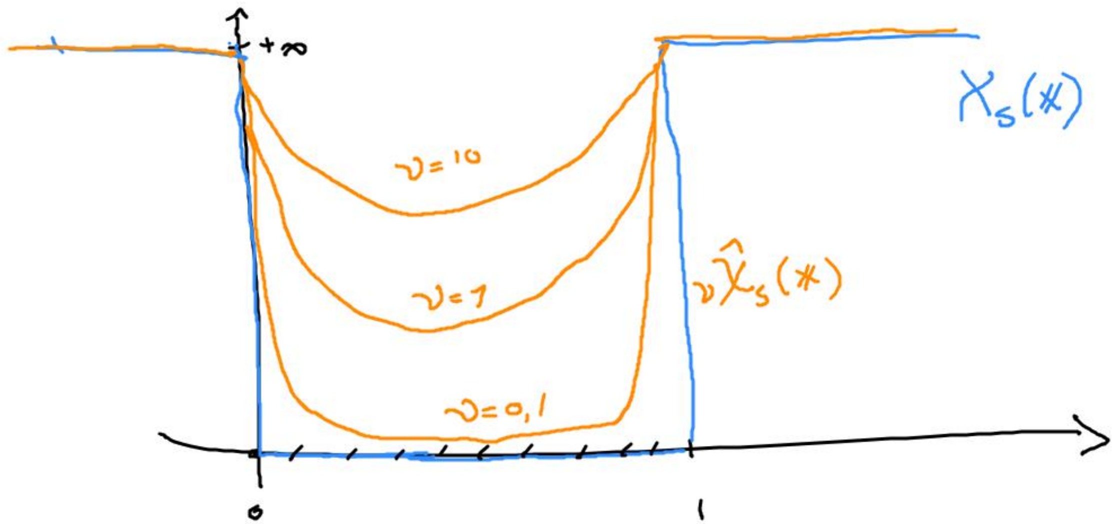
- ASSUME $\exists\, \bar{x} \in \mathbb{R}^n : g_i(\bar{x}) < 0,\ i=1,\dots,m$ [INTERIOR POINT SHOULD EXIST]

- IDEA: APPROXIMATE INDICATOR FUNCTION FROM ABOVE.

$$\chi_S(x) \approx \nu \hat{\chi}_S(x) = \begin{cases} \nu \sum_{i=1}^{m} \phi(g_i(x)) & \text{IF } g_i(x) < 0,\ i=1,\dots,m \\ +\infty & \text{OTHERWISE} \end{cases}$$

- ϕ PENALTY FUNCTION: $\mathbb{R}_- \to \mathbb{R}_+$, CONTINUOUS, $\lim_{s_k<0,\, s_k \to 0_-} \phi(s_k) = +\infty$
- EX: $\phi_1(s) = -\frac{1}{s}$, $\phi_2(s) = -\log(\min(1,-s))$

$$S = \begin{cases} x \le 1 \\ x \geq 0 \end{cases}$$



- NOTE: $\nu\hat{\chi}_s(x) \geq \chi_s(x)$ [UPPER BOUND]
- IF $\nu \to 0$, THEN $\nu\hat{\chi}_s(x) \to \chi_s(x)$
- <u>CONVERGENCE</u>: IF $x^*_{\nu_k}$ ARE OPTIMAL SOLUTIONS TO $\min f(x) + \nu_k \hat{\chi}_s(x)$ AND $\nu_k \to 0$, THEN $x^*_{\nu_k} \to x^*$ WHICH IS OPTIMAL IN (P)

USEFUL CONVERGENCE RESULTS

THEOREM: LET f, g_i AND $\phi \in C^1$ AND THAT $\phi'(s) \geq 0$ FOR $s < 0$. CONSIDER A SEQUENCE $\nu_k \to 0$. THEN

- x_k IS STATIONARY IN $\min f(x) + \nu_k \hat{\chi}_g(x)$
- $x_k \to \bar{x}$ AS $k \to \infty$
- LICQ HOLDS AT $\bar{x}$

$\Rightarrow$ $\bar{x}$ IS A KKT POINT TO THE ORIGINAL PROBLEM.

NOTE: IT IS POSSIBLE TO RETRIEVE THE KKT (LAGRANGE) MULTIPLIERS FOR $\bar{x}$ FROM THE METHOD

IF $\phi(s) = -\frac{1}{s}$ $\left\{ \nu_k / g_i^2(x_k) \right\} \longrightarrow \hat{\mu}_i$

RECAP: INTERIOR POINT METHOD

- TRANSFORM CONSTRAINED PROBLEM TO A SEQUENCE OF UNCONSTRAINED PROBLEMS.

- APPROXIMATING THE INDICATOR FUNCTION FROM ABOVE

- $\nu_k \to 0$, THEN $x^*_{\nu_k} \to x^*$

- $\nu_k \to 0$, x_k IS STATIONARY IN $\min f(x) + \nu_k \hat{\chi}_S(x)$, THEN $\{x_k\} \to \bar{x}$ KKT POINT IN ORIGINAL PROBLEM

- ν_k SHOULD BE DECREASED SLOWLY (SMOOTHLY) AS IN THE EXTERNAL CASE.