Lecture 14

Constrained optimization

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- Consider the optimization problem to

$$\begin{aligned} &\text{minimize} \;\; f(x), \\ &\text{subject to} \;\; x \in S, \end{aligned} \tag{1}$$

where $S \subset \mathbb{R}^n$ is non-empty, closed, and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable

- Basic idea behind all penalty methods: to replace the problem (1) with the equivalent unconstrained one:

$$\text{minimize} \;\; f(x) + \chi_S(x),$$

where

$$\chi_S(x) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise} \end{cases}$$

is the *indicator function* of the set S

- Feasibility is top priority; only when achieving feasibility can we concentrate on minimizing f
- Computationally bad: non-differentiable, discontinuous, and even not finite (though it is convex provided S is convex).
- Better: numerical “warning” before becoming infeasible or near-infeasible
- Approximate the indicator function with a numerically better behaving function

SUMT (Sequential Unconstrained Minimization Techniques)

- Suppose

$$S = \{\, x \in \mathbb{R}^n \mid g_i(x) \le 0, \quad i = 1, \dots, m, \\ h_j(x) = 0, \quad j = 1, \dots, \ell \,\},$$

- Choose C^0 penalty function $\psi : \mathbb{R} \to \mathbb{R}_+$ s.t. $\psi(s) = 0 \iff s = 0$
  - Typical choices: $\psi_1(s) = |s|$, or $\psi_2(s) = s^2$
- Approximate indicator function as

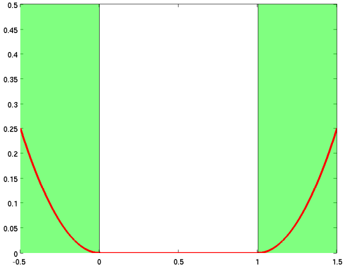
$$\chi_S(x) \approx \nu \check{\chi}_S(x) := \nu \left(\sum_{i=1}^{m} \psi\big(\max\{0, g_i(x)\}\big) + \sum_{j=1}^{\ell} \psi\big(h_j(x)\big) \right)$$

- $S = \{x \mid -x \leq 0, x \leq 1\}$

- Indicator function

$$\chi_S(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

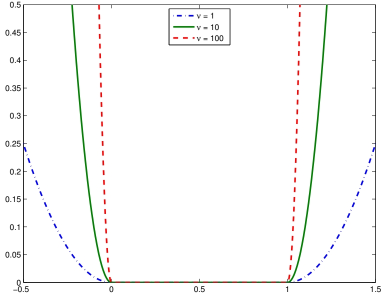
- $\nu\check{\chi}_S$ approximates χ_S from *below* ($\nu\check{\chi}_S \leq \chi_S$)



  - Penalty function $\psi(s) = s^2$
  - Approximate function (i.e. substitute for indicator function)

$$\nu\check{\chi}_S = \nu\Big((\max\{0, x-1\})^2 + (\max\{0, -x\})^2\Big)$$

- $\nu > 0$ is penalty parameter
- $\nu\check{\chi}_S(x) \to \chi_S(x)$ as $\nu \to \infty$.



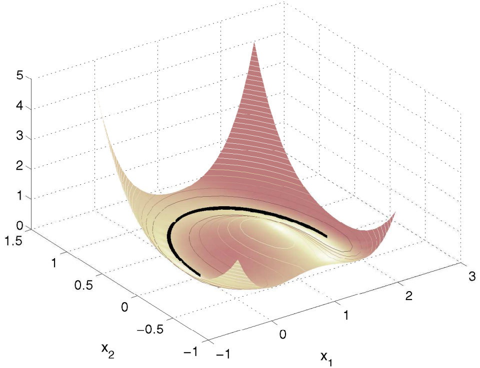
- Approximate function (i.e. substitute for indicator function)

$$\nu\check{\chi}_S = \nu\Big((\max\{0, x-1\})^2 + (\max\{0, -x\})^2\Big)$$

- Let $S = \{ x \in \mathbb{R}^2 \mid -x_2 \leq 0, (x_1 - 1)^2 + x_2^2 = 1 \}$
- Let $\psi(s) = s^2$. Then,

$$\check{\chi}_S(x) = [\max\{0, -x_2\}]^2 + [(x_1 - 1)^2 + x_2^2 - 1]^2$$

- Graph of $\check{\chi}_S$ and S:



- Consider increasing sequence $\{\nu_k\}$ with $\lim_{k\to\infty} \nu_k = \infty$
- Corresponding to a sequence of approximate problems

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) + \nu\check{\chi}_S(x) \tag{2}$$

  with optimal solutions $x^*_{\nu_k}$
- If $\{x^*_{\nu_k}\}$ has limit point $\hat{x}$, then $\hat{x}$ optimal solution to (1)

- Let x^* be optimal solution to

$$\operatorname*{minimize}_{x \in S} \quad f(x) \tag{1}$$

- For any $\nu > 0$, let x_ν^* be optimal solution to

$$\operatorname*{minimize}_{x \in \mathbb{R}^n} \quad f(x) + \nu \check{\chi}_S(x) \tag{2}$$

with $\check{\chi}_S(x) = \sum_{i=1}^m \psi\big(\max\{0, g_i(x)\}\big) + \sum_{j=1}^\ell \psi\big(h_j(x)\big)$

- Lower bound on $f(x^*)$

$$\forall \nu > 0, \quad f(x_\nu^*) + \nu \check{\chi}_S(x_\nu^*) \le f(x^*) + \nu \check{\chi}_S(x^*) \overset{\check{\chi}_S(x^*)=0}{=} f(x^*)$$

- (1) convex + $\psi(\cdot)$ convex + $\psi(s) \nearrow$ for $s \ge 0$ $\implies$ (2) convex

- Assume global optimal solution exists in original problem

$$\underset{x \in S}{\text{minimize}} \quad f(x) \tag{1}$$

- For any $\nu > 0$, assume x_ν^* global optimal solution exists for

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + \nu \check{\chi}_S(x) \tag{2}$$

Then, $\hat{x}$ limit point of $\{x_\nu^*\}$ as $\nu \to \infty \implies \hat{x}$ optimal to (1)

- Statement concerns **global optimal solutions** to (1) and (2)
- Statement useful if and only if (2) convex

- Let f, g_i $(i = 1, \ldots, m)$, and h_j $(j = 1, \ldots, \ell)$, be in C^1

Assume that the penalty function ψ is in C^1 and that $\psi'(s) \geq 0$ for all $s \geq 0$. Consider a sequence $\nu_k \to \infty$.

$$\left.\begin{array}{r} x_k \textbf{ stationary} \text{ in (2) with } \nu_k \\ x_k \to \hat{x} \text{ as } k \to +\infty \\ \text{LICQ holds at } \hat{x} \\ \hat{x} \text{ feasible in (1)} \end{array}\right\} \Longrightarrow \hat{x} \textbf{ stationary} \text{ (KKT) in (1)}$$

- From the proof we obtain estimates of Lagrange multipliers: the optimality conditions of (2) gives that

$$\mu_i^* \approx \nu_k \psi'[\max\{0, g_i(x_k)\}] \quad \text{and} \quad \lambda_j^* \approx \nu_k \psi'[h_j(x_k)]$$

- ν large $\implies f(x) + \nu\check{\chi}_S(x)$ difficult to minimize (cf. indicator function)
- If we **increase ν slowly** a good guess is that $x^*_{\nu_k} \approx x^*_{\nu_{k-1}}$.
- This guess can be improved.

- Consider **inequality** constrained optimization

$$\underset{x \in S}{\text{minimize}} \; f(x) \quad \text{with } S = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \; i = 1, \ldots, m \} \quad (1)$$

- Assume *strictly feasible* point exists: $\hat{x} \in \mathbb{R}^n$ s.t. $g_i(\hat{x}) < 0$ for all i
- **Interior penalty (barrier)** method approximates S from inside
- If a globally optimal solution to (1) is on the boundary of the feasible region, the method generates a sequence of interior points that converge to it

- Approximate χ_S from above

$$\chi_S(x) \leq \nu\hat{\chi}_S(x) := \begin{cases} \nu \sum_{i=1}^m \phi[g_i(x)], & \text{if } g_i(x) < 0,\ \forall\, i, \\ +\infty, & \text{otherwise,} \end{cases}$$

- $\phi : \mathbb{R}_- \to \mathbb{R}_+$, continuous, $\displaystyle\lim_{s_k<0,\ s_k\to 0_-} \phi(s_k) = \infty$
  - Typical examples: $\phi_1(s) = -s^{-1}$; $\phi_2(s) = -\log[\min\{1, -s\}]$
- The differentiable *logarithmic barrier function* $\widetilde{\phi}_2(s) = -\log(-s)$
  - $\widetilde{\phi}_2(s) < 0$ if $s < -1$, but same convergence theory
- g_i convex $+$ ϕ convex $+$ $\phi \nearrow$ for $s < 0 \implies \nu\hat{\chi}_S$ convex

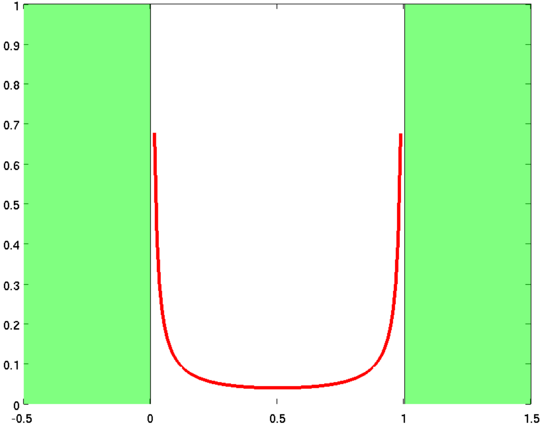
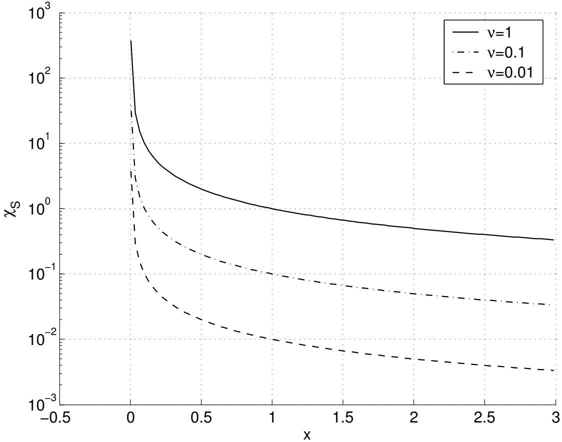


Figure: Feasible set is $S = \{x \mid -x \leq 0, x \leq 1\}$. Barrier function $\phi(s) = -1/s$, barrier parameter $\nu = 0.01$.

Consider $S = \{ x \in \mathbb{R} \mid -x \leq 0 \}$. Choose $\phi = \phi_1 = -s^{-1}$. Graph of the barrier function $\nu\hat{\chi}_S$ in below figure for various values of ν (note how $\nu\hat{\chi}_S$ converges to χ_S as $\nu \downarrow 0$!):



- Penalty problem:

$$\text{minimize } f(x) + \nu \hat{\chi}_S(x) \tag{2}$$

- **Global optimal solutions** to (2) $\to$ global optimal solution to (1)
- Convergence of stationary points also holds:

> Let f and g_i ($i = 1, \ldots, m$), an ϕ be in C^1, and that $\phi'(s) \geq 0$ for all $s < 0$. Consider sequence $\nu_k \to 0$. Then:
>
> $$\left.\begin{array}{r} x_k \text{ **stationary** in (3) with } \nu_k \\ x_k \to \hat{x} \text{ as } k \to +\infty \\ \text{LICQ holds at } \hat{x} \end{array}\right\} \implies \hat{x} \text{ **stationary** (KKT) in (1)}$$

- $\phi(s) = \phi_1(s) = -1/s$, then $\phi'(s) = 1/s^2 \implies \{\nu_k / g_i^2(x_k)\} \to \hat{\mu}_i$.

- Consider the LP

$$
\begin{aligned}
\text{minimize} \quad & -b^T y, \\
\text{subject to} \quad & A^T y + s = c, \\
& s \geq 0^n,
\end{aligned}
\tag{3}
$$

and the corresponding KKT conditions:

$$
\begin{aligned}
A^T y + s &= c, \\
Ax &= b, \\
x \geq 0^n,\ s \geq 0^n,\ x^T s &= 0
\end{aligned}
\tag{4}
$$

- Apply barrier method for (3), taking care of $s \geq 0$. Subproblem:

$$\begin{aligned} \text{minimize} \quad & -b^T y - \nu \sum_{j=1}^{n} \log(s_j) \\ \text{subject to} \quad & A^T y + s = c \end{aligned}$$

- The KKT conditions for subproblem:

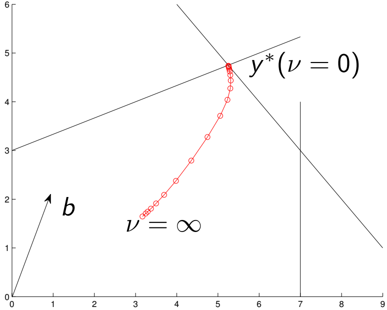
$$\begin{aligned} A^T y + s &= c, \\ Ax &= b, \\ x_j s_j &= \nu, \quad j = 1, \dots, n \end{aligned} \tag{5}$$

- (5): (4) with complementary slackness perturbed by ν

Optimal solutions to subproblems

$$\text{minimize } -b^T y - \nu \sum_{j=1}^{n} \log(s_j)$$
$$\text{subject to } A^T y + s = c$$

for different ν's form the central path.



Consider problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \leq \mathbf{0} \\ & h(x) = \mathbf{0} \end{aligned}$$

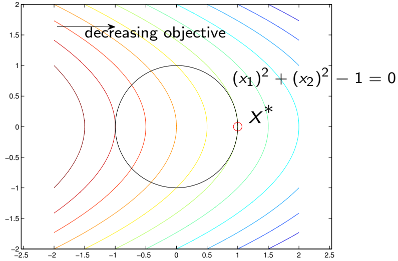
- We have good solution methods for quadratic programs (QP) (e.g., simplicial decomposition and gradient projection method)
- At iterate x_k, approximate original problem with QP subproblem. Find search direction p by solving QP subproblem

$$\begin{aligned} \underset{p}{\text{minimize}} \quad & \tfrac{1}{2} p^T \nabla^2 f(x_k) p + \nabla f(x_k)^T p \\ \text{subject to} \quad & g_i(x_k) + \nabla g_i(x_k)^T p \leq 0, \quad i = 1, \ldots, m \\ & h_j(x_k) + \nabla h_j(x_k)^T p = 0, \quad j = 1, \ldots, l \end{aligned}$$

- Suggested method does not always work!

Consider problem

$$\min_{x} \; -x_1 - \tfrac{1}{2}(x_2)^2$$
$$\text{s.t.} \;\; (x_1)^2 + (x_2)^2 - 1 = 0$$



Optimal solution $(1,0)^T$, consider QP subproblem at $x_1 = 1.1$, $x_2 = 0$:

$$\begin{aligned} \underset{p}{\text{minimize}} \quad & -p_1 - \tfrac{1}{2}(p_2)^2 \\ \text{subject to} \quad & p_1 + 0.0955 = 0 \end{aligned}$$

QP subproblem unbounded – bad linear approx. of nonlinear constraint!

- Linearized constraints might be too inaccurate!
- Account for nonlinear constraints in objective – Lagrangian idea.

$$L(x, \mu_k, \lambda_k) = f(x) + \mu_k^T g(x) + \lambda_k^T h(x).$$

- Solve (improved) QP subproblem to find search direction p:

$$\begin{aligned} \underset{p}{\text{minimize}} \quad & \tfrac{1}{2} p^T \nabla^2_{xx} L(x_k, \mu_k, \lambda_k) p + \nabla f(x_k)^T p \\ \text{subject to} \quad & g_i(x_k) + \nabla g_i(x_k)^T p \leq 0, \quad i = 1, \ldots, m \\ & h_j(x_k) + \nabla h_j(x_k)^T p = 0, \quad j = 1, \ldots, l \end{aligned}$$

- Direction p, with multipliers μ_{k+1}, λ_{k+1}, define Newton step for solving (nonlinear) KKT conditions (see text for more).
- Lagrangian Hessian $\nabla^2_{xx} L(x_k, \mu_k, \lambda_k)$ may not be positive definite.

- Given $x_k \in \mathbb{R}^n$ and a vector $(\mu_k, \lambda_k) \in \mathbb{R}^m_+ \times \mathbb{R}^\ell$, choose a positive definite matrix $B_k \in \mathbb{R}^{n\times n}$. $B_k \approx \nabla^2_{xx} L(x_k, \mu_k, \lambda_k)$

- Solve

$$\begin{aligned} \underset{p}{\text{minimize}} \quad & \frac{1}{2} p^T B_k p + \nabla f(x_k)^T p, && \text{(6a)} \\ \text{subject to} \quad & g_i(x_k) + \nabla g_i(x_k)^T p \le 0, \; i = 1, \dots, m, && \text{(6b)} \\ & h_j(x_k) + \nabla h_j(x_k)^T p = 0, \; j = 1, \dots, \ell && \text{(6c)} \end{aligned}$$

- Working version of SQP search direction subproblem
- Quadratic convergence near KKT points. What about global convergence? Perform line search with some merit function.

1. Initialize iterate with (x_0, μ_0, λ_0), B_0 and merit function M.
2. At iteration k with (x_k, μ_k, λ_k) and B_k, solve QP subproblem for search direction p_k:

$$\begin{aligned} \underset{p}{\text{minimize}} \quad & \tfrac{1}{2} p^T B_k p + \nabla f(x_k)^T p \\ \text{subject to} \quad & g_i(x_k) + \nabla g_i(x_k)^T p \leq 0, \quad i = 1, \ldots, m \\ & h_j(x_k) + \nabla h_j(x_k)^T p = 0, \quad j = 1, \ldots, l \end{aligned}$$

   Let μ_k^* and λ_k^* be optimal multipliers of QP subproblem. Define $\Delta x = p_k$, $\Delta \mu = \mu_k^* - \mu_k$, $\Delta \lambda = \lambda_k^* - \lambda_k$.
3. Perform line search to find $\alpha_k > 0$ s.t. $M(x_k + \alpha_k \Delta x) < M(x_k)$.
4. Update iterates:
   $x_{k+1} = x_k + \alpha_k \Delta x$, $\mu_{k+1} = \mu_k + \alpha_k \Delta \mu$, $\lambda_{k+1} = \lambda_k + \alpha_k \Delta \lambda$.
5. Stop if converge, otherwise update B_k to B_{k+1}; go to step 2.

Merit function as *non-differentiable* exact penalty function P_e:

$$\check{\chi}_S(x) := \sum_{i=1}^{m} \text{maximum}\,\{0, g_i(x)\} + \sum_{j=1}^{\ell} |h_j(x)|,$$
$$P_e(x) := f(x) + \nu\check{\chi}_S(x)$$

- For large enough ν, solution to QP subproblem (6) defines a descent direction for P_e at (x_k, μ_k, λ_k).
- For large enough ν, reduction in P_e implies progress towards KKT point in the original constrained optimization problem.
  - Compare convergence results for exterior penalty methods.
  - See text for more (Proposition 13.10).

*Convergence of merit SQP method

- Combining the descent direction property and exact penalty function property, one can prove convergence of the merit SQP method.

- Convergence of the SQP method towards KKT points can be established under additional conditions on the choices of matrices $\{B_k\}$
  1. Matrices B_k bounded
  2. Every limit point of $\{B_k\}$ positive definite

- Selecting the value of ν is difficult
- No guarantees that the subproblems (6) are feasible; we *assumed* above that the problem is well-defined
- P_e is only continuous; some step length rules infeasible
- Fast convergence not guaranteed (the *Maratos effect*)
- Penalty methods in general suffer from ill-conditioning. For some problems the ill-conditioning is avoided
- Exact penalty SQP methods suffer less from ill-conditioning, and the number of QP:s needed can be small. They can, however, cost a lot computationally
- `fmincon` in MATLAB is an SQP-based solver