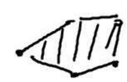
LECTURE 2: CONVEXITY

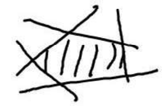
- A SET $S \subseteq \mathbb{R}^n$ IS CONVEX IF $\left.\begin{array}{l} x_1, x_2 \in S \\ \lambda \in (0,1) \end{array}\right\} \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$
- THE INTERSECTION OF CONVEX SETS IS A CONVEX SET
- LET $S = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ i=1,\dots,m \}$. THEN IF g_i ARE CONVEX FUNCTIONS, THEN S IS A CONVEX SET.

conv S

- THE CONVEX HULL OF A SET S IS THE SET OF ALL CONVEX COMBINATIONS OF POINTS IN S
- POLYTOPE = THE CONVEX HULL OF FINITELY MANY POINTS



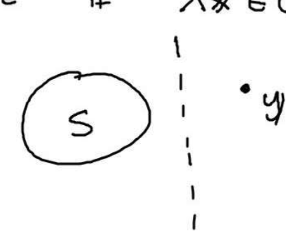
- POLYHEDRON = THE INTERSECTION OF FINITELY MANY HALF-SPACES



- REPRESENTATION THM: "POLYHEDRON = POLYTOPE + POLYHEDRAL CONE"

- "EXTREME POINT TO A CONVEX SET": A POINT THAT CANNOT BE WRITTEN AS A CONV COMB. OF TWO OTHER POINTS 

- A SET $C \subseteq \mathbb{R}^n$ IS A CONE IF $\lambda x \in C$ WHENEVER $x \in C$ AND $\lambda > 0$ 

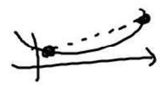
- SEPARATION THM: 

- FARKAS' LEMMA: EXACTLY ONE OF THE SYSTEMS

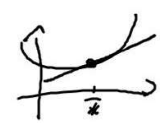
$$\begin{cases} Ax = b \\ x \geq 0 \end{cases} \qquad \begin{cases} A^T y \leq 0 \\ b^T y > 0 \end{cases}$$

↖ VERIFY INEQ ☺

CONVEX FUNCTIONS



- A FUNCTION f IS CONVEX IF $f(\lambda x^1 + (1-\lambda)x^2) \le \lambda f(x^1) + (1-\lambda) f(x^2)$
- A FUNCTION f IS CONCAVE IF $-f$ IS CONVEX
- THE SUM OF CONVEX FUNCTIONS IS A CONVEX FUNCTION.
- THE EPIGRAPH OF A FUNCTION f IS CONVEX $\iff$ f IS CONVEX
- IF $f \in C^1$: THEN f IS CONVEX IF $f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x})$



- IF $f \in C^2$: —— " —— IF $\nabla^2 f(x) \succeq 0$ FOR ALL x

CONVEX PROBLEM

- A PROBLEM $\begin{cases} \min f(x) \\ \text{s.t. } x \in S \end{cases}$ IS A CONVEX PROBLEM IF f IS CONVEX ON S AND S IS CONVEX

LECTURE 3: OPTIMALITY CONDITIONS

CONSIDER (P) $\min f(x)$
s.t. $x \in S$

- GLOBAL MINIMA –
- LOCAL MINIMA –
- FUNDAMENTAL THM OF OPTIMALITY: IF (P) IS A CONVEX PROBLEM THEN EVERY LOCAL MIN IS A GLOBAL MIN.
- WEIERSTRASS' THEOREM:

$$\left.\begin{array}{l} - S \text{ NONEMPTY, CLOSED} \\ - f \text{ WEAKLY COERCIVE W.R.T. } S \\ - f \text{ LOWER SEMI-CONT} \end{array}\right\} \Rightarrow \exists \text{ GLOBAL MIN TO (P).}$$

UNCONSTRAINED PROBLEMS

$$\min \ f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n$$

- IF $f \in C^1$: x^* LOCAL MIN $\Rightarrow$ $\nabla f(x^*) = 0$ [PROOF IS SIMPLE AND USEFUL]
- IF $f \in C^2$: — " — $\Rightarrow$ $\begin{cases} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) \succeq 0 \end{cases}$
- IF f IS CONVEX: x^* GLOBAL MIN $\Leftrightarrow$ $\nabla f(x^*) = 0$

CONSTRAINED PROBLEMS

$$\min f(x) \quad \text{s.t. } x \in S$$

- IF x^* IS A LOCAL MIN $\Rightarrow$

EMILS OPT. COND.

"IT SHOULD NOT BE POSSIBLE TO FIND A DIRECTION AT x^* WHICH IS BOTH A FEASIBLE AND A DESCENT DIRECTION".

$\Rightarrow \nabla f(x^*)^T p \geq 0$ FOR ALL FEASIBLE DIRECTIONS p.

- IF S IS A CONVEX SET.

x^* LOCAL MIN $\Rightarrow x^*$ STATIONARY

a) $\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in S$

b) $\min_{x \in S} \nabla f(x^*)^T (x - x^*) = 0$

c) $x^* = \text{Proj}_S (x^* - \nabla f(x^*))$

d) $-\nabla f(x^*) \in N_S(x^*)$

LECTURE 4: UNCONSTRAINED OPT. METHODS

- LINE SEARCH TYPE ALGORITHMS
  - 0 INITIALIZE
  - 1 FIND DESCENT DIR
  - 2 FIND STEP LENGTH
  - 3 UPDATE

- SEARCH DIRECTION
  - STEEPEST DESCENT: $p_k = -\nabla f(x_k)$
  - NEWTON $\nabla^2 f(x_k) p_k = -\nabla f(x_k)$
  - LEWENBERG-MARQUARDT $\left[\nabla^2 f(x_k) + \gamma I\right] p_k = -\nabla f(x_k)$

- STEP LENGTS
  - ARMIJOS RULE
  - EXACT

LECTURE 5/6: OPTIMALITY CONDITIONS

$$\min f(x) \quad \text{s.t. } x \in S$$

GEOMETRIC OPTIMALITY CONDITIONS:

- EMILS OPT COND:

x^* LOCAL MIN $\Rightarrow$ "SHOULD NOT EXIST VECTORS p WHICH ARE BOTH DESCENT/FEASIBLE"

$$\Rightarrow A(x^*) \cap B(x^*) = \emptyset$$

FEASIBLE DIRECTIONS — DESCENT DIRECTIONS

$\mathring{F}(x^*) = \{p : \nabla f(x^*)^T p < 0\}$

TANGENT CONE

$$T_S(x) = \{ p \mid \exists \{x_k\} \in S, \{\lambda_k\} \geq 0 : \lim x_k = x, \lim \lambda_k(x_k - x) = p \}$$

GEOMETRIC OPT COND:

x^* LOCAL MIN $\Longrightarrow$ $\mathring{F}(x^*) \cap T_S(x^*) = \emptyset$

KKT CONDITIONS

- ASSUME $S = \{x \mid g_i(x) \leq 0,\ i=1,\dots,m\}$
- GRADIENT CONE: $G(x) = \{p \mid \nabla g_i(x)^T p \leq 0,\ i \in I(x)\}$
- $T_S(x) \subseteq G(x)$
- ABADIE'S CQ: $T_S(x) = G(x)$
- ASSUMING ABADIE'S CQ $\Longrightarrow$ x^* LOCAL MIN $\Rightarrow$ $\mathring{F}(x^*) \cap G(x^*) = \emptyset$

$\Rightarrow$ x^* LOCAL MIN $\Rightarrow$ $\mathring{F}(x^*) \cap G(x^*) = \emptyset$

- THIS CAN BE FORMULATED ~~AS~~ AN INCONSISTENT INEQ SYSTEM

x^* LOCAL MIN $\Rightarrow$ $\begin{cases} \nabla f(x^*)^T p < 0 \\ \nabla g_i(x^*)^T p \leq 0, \quad i \in I(x) \end{cases}$ INCONSISTENT

$\Rightarrow$ [FARKAS' LEMMA]

$\Rightarrow$ KKT SYSTE $\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0 \\ \mu_i g_i(x^*) = 0, \quad i=1,\dots,m \\ g_i(x) \leq 0, \quad i=1,\dots,m \\ \mu_i \geq 0, \quad i=1,\dots,m \end{cases}$ SOLVABLE

– IF ABADIE'S CQ HOLDS:

[NECESSARY] x^* LOCAL MIN $\Rightarrow$ x^* KKT POINT

– IF THE PROBLEM IS CONVEX

[SUFFICIENT] x^* KKT POINT $\Rightarrow$ x^* GLOBAL MIN

LECTURE 7: LAGRANGIAN DUALITY.

- $f^* = \inf f(x)$
  s.t. $g_i(x) \leq 0$, $i=1,\dots,m$
  $x \in X$

- g_i - CONSTRAINTS ARE COMPLICATED $\Rightarrow$ LIFT THEM TO THE OBJECTIVE WITH A PENALTY

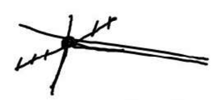
- LAGRANGIAN DUAL FUNCTION: ALWAYS CONCAVE $q(\mu) = \min_{x \in X} f(x) + \mu^T g(x)$, DUAL PROBLEM $q^* = \sup q(\mu)$ s.t. $\mu \geq 0$

- WEAK DUALITY: FOR ANY FEASIBLE x AND ANY $\mu \geq 0$:
  $$q(\mu) \leq f(x)$$

- STRONG DUALITY: IF SLATER'S CQ HOLDS AND f IS CONVEX, THEN
  $$f^* = q^*$$

LECTURE 8/9 LINEAR PROGRAMMING

- MINIMIZE A LINEAR FUNCTION OVER A POLYHEDRON
- IF AN OPTIMAL SOLUTION EXISTS $\Rightarrow$ ONE OF THE EXTREME POINTS MUST BE OPTIMAL
- STANDARD FORM: $\begin{cases} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$
- EXTREME POINTS $\Longleftrightarrow$ BASIC FEASIBLE SOLUTIONS (BFS)
- BFS: PARTITION $A = [B, N]$ SUCH THAT $\text{rank } B = m$ AND $B^{-1} b \geq 0$
- DEGENERATE BFS: $B^{-1} b \not> 0$



- SIMPLEX METHOD: "ITERATIVELY UPDATES PARTITION $A = [B, N]$ BY REPLACING ONE COLUMN IN B WITH A COLUMN FROM N"

- PHASE I - SIMPLEX
  - FIND INITIAL BFS.
  - $$(P)\ \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad \xRightarrow{\text{INTRODUCE VAR. } a} \quad (P')\ \begin{array}{ll} \min & \overbrace{\mathbb{1}^T a}^{\sum_{i=1}^{m} a_i} \\ \text{s.t.} & Ax + a = b \\ & x \geq 0 \\ & a \geq 0 \end{array}$$

    $\tilde{A} = [A, I]$

    $\hat{A}\begin{bmatrix} x \\ a \end{bmatrix} = b$
  - SOLVE (P') WITH SIMPLEX METHOD.
  - THIS CAN BE DONE SINCE USING $x_B = a$ IS A BFS

    $\Rightarrow A = [I, A]$
  - WHEN OPTIMAL SOLUTION FOR (P') IS FOUND, NO ARTIFICIAL VARIABLES IS INCLUDED IN THE BASIC VARIABLES
  - YOU NOW HAVE AN INITAL BFS TO (P)

LECTURE 10 LP DUALITY

PRIMAL | DUAL

- $\min \mathbf{c}^T \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ $\Longleftrightarrow$ $\max \mathbf{b}^T \mathbf{y}$ s.t. $A^T \mathbf{y} \leq \mathbf{c}$, $\mathbf{y} \in \mathbb{R}^m$ (FREE)

$$\left.\begin{array}{ll} \min & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} \max & \mathbf{b}^T \mathbf{y} \\ \text{s.t.} & A^T \mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \in \mathbb{R}^m \text{ (FREE)} \end{array}\right.$$

n VARIABLES | m VARIABLES

m CONSTRAINTS | n CONSTRAINTS

- WEAK DUALITY: FOR $\mathbf{x}$ AND $\mathbf{y}$ FEASIBLE IN THEIR PROBLEMS

$$\mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{y}$$

- STRONG DUALITY: IF BOTH PROBLEMS FEASIBLE, THEN

$$\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$$

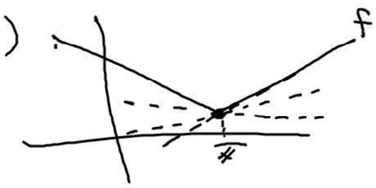
- COMPLEMENTARY SLACKNESS: "EITHER THE SLACK OR THE DUAL VARIABLE CORRESPONDING TO A CONSTRAINT IS 0"

- MEANING OF OPTIMAL DUAL VARIABLES:

  y_i^* IS HOW MUCH WILL THE OPTIMAL OBJECTIVE VALUE OF THE PRIMAL PROBLEM CHANGE IF b_i IS INCREASED WITH 1.

LECTURE 11: CONVEX OPTIMIZATION

- A SUBGRADIENT $\mathbb{p}$ TO A CONVEX FUNCTION f AT $\bar{x}$ IS A VECTOR FULFILLING: $f(x) \geq f(\bar{x}) + \mathbb{p}^T(x - \bar{x})$



- THEY ALWAYS EXIST FOR CONVEX FUNCTIONS
- IF f IS DIFFERENTIABLE AT $\bar{x}$, THEN

$$\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$$

- SUBGRADIENT METHOD: $x_{k+1} = x_k - \alpha_k \underbrace{\mathbb{p}_k}_{\text{SUBGRADIENT}}$

LECTURE 13 FEASIBLE DIRECTION METHODS

- ITERATIVE METHODS
  - 0: INITIALIZE
  - 1: FIND SEARCH DIRECTION WHICH IS FEASIBLE
  - 2: STEP LENGTH
  - 3: UPDATE

- FRANK-WOLFE : LINEARIZE PROBLEM IN EACH ITERATION AND SOLVE LP.
  [S POLYHEDRON]

- SIMPLICAL DECOMPOSITION : SIMILAR TO F-W. BUT REMEMBERS VISITED EXTREME POINTS

- GRADIENT PROJECTION ALG: $x_{k+1} = \text{Proj}_S \left[x_k - \alpha_k \nabla f(x_k) \right]$
  [S EASY TO PROJECT TO]

LECTURE 14 CONSTRAINED OPT. METHODS

- $\begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array} \Bigg\} \implies \min_{x \in \mathbb{R}^n} f(x) + \chi_S(x)$ $\qquad \chi_S(x) = \begin{cases} 0 & \text{IF } x \in S \\ +\infty & \text{OTHERWISE} \end{cases}$

- EXTERIOR PENALTY METHOD [PENALIZE BEING INFEASIBLE]
  - APPROXIMATE INDICATOR FUNCTION FROM BELOW $\sim \check{\chi}_S(x) \leq \chi_S(x)$
- INTERIOR PENALTY METHOD [PENALIZE BEING CLOSE TO CONSTRAINTS]
  - APPROXIMATE —— " " —— FROM ABOVE $\sim \hat{\chi}_S(x) \geq \chi_S(x)$