A theorem with a way to check d-separation

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Theorem 1. *Given a Bayesian Network and disjoint sets of nodes X, Y, and Z. Perform the following steps:*

1. *Remove all links from nodes in Z to their children.*

2. *Repeatedly, remove all nodes not in X, Y, or Z that do not have any children.*

Then X and Y are d-separated given Z in the initial network if and only if, in the new network, there are no trails from X to Y.

We first prove a sequence of lemmas:

Lemma 1. *Step 1 above does not change whether X and Y are d-sepatated given Z.*

Proof. It is enough to prove this for a single node in z_1 in Z and a link to a single one of its children; let us call this link L. Let's assume there exists an active trail from X to Y in the network where L is deleted. Then the same trail will exist in the initial network, at it is easy to see that it will also be active there.

Let us instead assume there is an active trail from X to Y in the initial network. Such a trail cannot include the link L, as it would otherwise have been blocked by z_1. Thus the trail will also exist in the network where L is removed. It will also be active in the reduced network: Outside of v-structures, it cannot be blocked, or it would have been blocked already in the full network. In a v-structure, we know that in the intial network there is a path from the v-structure node to a node in Z. However, if this path is disrupted by removing L, there will already be a path from the v-structure node to z_1. So the trail is active in the reduced network.

In other words, an active trail from X and Y exists in the full network if and only if an active trail from X to Y exists in the reduced network, and the lemma is proved. □

Lemma 2. *Step 2 above does not change whether X and Y are d-separated given Z.*

Proof. It is enough to prove this when a single childless v node is removed. Assume there is an active trail from X to Y in the reduced network. Then this trail will exist also in the original network, and it is easy to see that it must be active.

If we instead assume there is an active trail from X to Y in the reduced network, then this trail cannot contain v: Otherwise it would have been blocked by v. Thus the trail exists also in the reduced network, and it is easy to see that it must be active there.

In other words, and active trail from X to Y exists in the full network if and only if an active trail from X to Y exists in the reduced network, and the lemma is proved. □

Lemma 3. *A network where the two steps in the theorem have been performed has the following properties:*

1. *All nodes not in X, Y, or Z have a descendant in either X, Y, or Z.*
2. *Nodes in Z have no descendants.*

Proof. Assume that a node v not in X, Y, or Z does not have any descendants in X, Y, or Z. If v has no descendants, it would have been removed in Step 2, so that is impossible. Furthermore, let w denote the descendant with the maximum number of generations separating v and w. Then w has no descendants and is not in X, Y, or Z. This, too, violates the assumption that Step 2 has been performed.

The second condition follows directly. □

Lemma 4. *Assume a Bayesian Network fulfills the conditions of the lemma above. If it contains any trails from X to Y it will also contain an active trail from X to Y.*

Proof. Generally, a trail can be blocked either outside v-structures, or in a v-structure. In a network fulfilling condtion 2 above, no trail can be blocked outside a v-structure, as that would require some node in Z to have a descendant.

Now assume that the network contains an inactive trail blocked at n v-structures. We prove that there then exists a trail blocked at $n-1$ v-structures: Take a trail blocked at n v-structures and let v be a node at one of these v-structures. Then v has no descendants in Z, so according to the first condition above, it must have a descendant in either X or Y. If it has a descendant in X, we see that we may construct a new trail from X to Y, using this descendancy, where v is not in a v-structure. The same is the case if v has a descendant in Y. Thus we may construct a trail with $n-1$ blocking v-structures.

By recursion we may now construct a trail that is blocked at no v-structures, and which is thus not blocked at all: It is an active trail. □

We can now put together a proof of the theorem: Assume X and Y are d-separated given Z. Using Lemma 1 and 2, we get that X and Y would also be d-separated in the network obtained by performing steps 1 and 2 of the theorem. According to Lemma 3, the reduced network has the properties listed in that Lemma. Now assumed the reduced network contained any trail from X to Y. Then according to Lemma 4 it would also contain an active trail, contradicting the assumption that X and Y are d-separated in this network. Thus no trail from X to Y exists.

Conversely, assume there is no trail from X to Y in the reduced network. Then surely there is no active trail, so X and Y are d-separated in the reduced network. Thus according to Lemmas 1 and 2 they are d-separated in the original network.