Formula sheet / Formelsamling
Mathematical statistics, TMA074

Probability basics

- The following holds for probabilities
  - ⋆ $0 \leq \mathsf{P}(A) \leq 1$
  - ⋆ $\mathsf{P}(\Omega) = 1$
  - ⋆ $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$ for disjoint events A and B
- Addition rule: $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$
- Conditional probability: $\mathsf{P}(B \mid A) = \dfrac{\mathsf{P}(A \cap B)}{\mathsf{P}(A)}$
- Bayes' formula: $\mathsf{P}(A \mid B) = \dfrac{\mathsf{P}(B \mid A)\mathsf{P}(A)}{\mathsf{P}(B)}$
- Law of total probability: $\mathsf{P}(A) = \sum\limits_{i=1}^{n} \mathsf{P}(A \mid B_i)\mathsf{P}(B_i)$, for a partition $\bigcup_{i=1}^{n} B_i = \Omega$ of Ω into disjoint events $B_1, \ldots, B_n$
- A and B are independent $\Leftrightarrow \mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B)$

Univariate random variables (r.v.)

- Distribution function of X: $F_X(x) = \mathsf{P}(X \leq x)$
- Probability mass function for discrete r.v. X: $f_X(k) = \mathsf{P}(X = k)$
- Density function for continuous r.v. X: $f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}$
- $\mathsf{P}(a < X \leq b) = F_X(b) - F_X(a) = \begin{cases} \sum_{k=a+1}^{b} f_X(k) & \text{(discrete r.v. with } a, b \text{ integer values)} \\ \int_a^b f_X(x)\,\mathrm{d}x & \text{(continuous r.v.)} \end{cases}$

Multivariate random variables

- Joint probability density:

$$F_{X,Y}(x,y) = \mathsf{P}(X \leq x, Y \leq y) = \begin{cases} \sum_{i \leq x, j \leq y} f_{X,Y}(i,j), & \text{(discrete r.v.)} \\ \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(t,u)\,\mathrm{d}t\,\mathrm{d}u, & \text{(continuous r.v.)} \end{cases}$$

- Marginal density:

$$f_X(x) = \begin{cases} \sum_{y=-\infty}^{\infty} f_{X,Y}(x,y) & \text{(discrete r.v.)} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y & \text{(continuous r.v.)} \end{cases}$$

- Conditional density: $f_{X|y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
- $\mathsf{P}(a \leq X \leq b \text{ and } c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$.
- X and Y are independent if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y.

Expectations

- Expectation of $g(X,Y)$:

$$\mathsf{E}(g(X,Y)) = \begin{cases} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g(i,j) f_{X,Y}(i,j), & \text{(discrete r.v.)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y, & \text{(continuous r.v.)} \end{cases}$$

- Variance: $\mathsf{V}(X) = \mathsf{E}((X - \mathsf{E}(X))^2) = \mathsf{E}(X^2) - \mathsf{E}(X)^2$
- Standard deviation: $\sqrt{V(X)}$
- Covariance: $\mathsf{C}(X,Y) = \mathsf{E}((X - \mathsf{E}(X))(Y - \mathsf{E}(Y))) = \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y)$
- Expectation of linear combinations: $\mathsf{E}(\sum_{i=1}^n a_i X_i + b) = \sum_{i=1}^n a_i \mathsf{E}(X_i) + b$
- Variance of linear combinations: $\mathsf{V}(\sum_{i=1}^n a_i X_i + b) = \sum_{i=1}^n a_i^2 \mathsf{V}(X_i) + 2\sum_{i=1}^n \sum_{j=i+1}^n a_i a_j \mathsf{C}(X_i, X_j)$
- Independence of r.v.'s $X_1, \ldots, X_n$ implies that they are uncorrelated $\mathsf{C}(X_i, X_j) = 0, i \neq j$
- Correlation coefficient: $\rho(X,Y) = \frac{\mathsf{C}(X,Y)}{\sqrt{\mathsf{V}(X)\mathsf{V}(Y)}}$

Properties of common distributions

- $X \sim \mathrm{Bin}(n_1, p)$, $Y \sim \mathrm{Bin}(n_2, p)$ independent $\Rightarrow X + Y \sim \mathrm{Bin}(n_1 + n_2, p)$
- $X \sim \mathrm{Po}(\mu_1)$, $Y \sim \mathrm{Po}(\mu_2)$ independent $\Rightarrow X + Y \sim \mathrm{Po}(\mu_1 + \mu_2)$
- $X \sim \mathsf{N}(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim \mathsf{N}(0,1)$
- $X \sim \mathsf{N}(\mu, \sigma^2) \Rightarrow F_X(x) = \Phi(\frac{x-\mu}{\sigma})$ where $\Phi(\cdot)$ given in table 1.
- $X_i \sim \mathsf{N}(\mu_i, \sigma_i^2), i = 1, \ldots, n$ independent $\Rightarrow \sum_{i=1}^n a_i X_i \sim \mathsf{N}(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2)$
- $X \sim \mathsf{N}(0,1), Y \sim \chi^2(\nu)$ independent $\Rightarrow \frac{X}{\sqrt{Y/\nu}} \sim t(\nu)$
- $X_1, \ldots, X_n$ independent and $\mathsf{N}(0,1) \Rightarrow \sum_{i=1}^n X_i^2 \sim \chi^2(n)$
- $X_1, \ldots, X_n$ independent and $\mathsf{N}(\mu, \sigma^2) \Rightarrow \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2(n-1)$
- $X \sim \chi^2(n), Y \sim \chi^2(m)$ independent $\Rightarrow \frac{X/n}{Y/m} \sim F(n,m)$
- $F_{1-\alpha}(n,m) = 1/F_\alpha(m,n)$

Central limit theorem

For $X_1, \ldots, X_n$ independent and identically distributed with $\mathsf{E}(X_i) = \mu_i$, $\mathsf{V}(X_i) = \sigma^2$ it holds $\sum_{i=1}^n X_i$ is approximately $\mathsf{N}(n\mu, n\sigma^2)$-distributed for n large enough. From this the following approximations follow:

- $\mathrm{Po}(\mu) \approx \mathsf{N}(\mu, \mu)$ for $\mu \geq 15$
- $\mathrm{Bin}(n,p) \approx \mathsf{N}(np, np(1-p))$ for $np(1-p) \geq 10$
- $\mathrm{Bin}(n,p) \approx \mathrm{Po}(np)$ for $p \leq 0.1$ and $n \geq 10$

Statistics and point estimates

Describing data:

- Sample mean/average: $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$
- Sample variance: $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 - n \cdot \bar{x}^2\right]$
- Sample covariance: $c_{xy} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$
- (Sample) correlation coefficient: $r_{xy} = \frac{c_{xy}}{s_x s_y}$

Let $x_1, \ldots, x_n$ be independent and identically distributed r.v. $X_1, \ldots, X_n$ with expectation μ and variance σ^2, then the sample mean is an unbiased estimator for μ and sample variance an unbiased estimator for σ^2.

Distribution	$f(x)$		Expectation	Variance
Binomial Bin(n,p)	$\binom{n}{k} p^k(1-p)^{n-k}$	$k=0,\dots,n$	np	np(1-p)
Negativ binomial nBin(r,p)	$\binom{k-1}{r-1} p^r(1-p)^{k-r}$	$k=r,r+1\dots$	r/p	$r(1-p)/p^2$
Hypergeometric Hyp(N,n,r)	$\dfrac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}$	$k=\max(0,n+r-N),\dots,\min(n,r)$	nr/N	$\frac{nr(N-r)(N-n)}{N^2(N-1)}$
Poisson Po(μ)	$p(k)=\frac{\mu^k}{k!}e^{-\mu}$	$k=0,1,\dots$	μ	μ
Geometric Ge(p)	$p(1-p)^{k-1}$	$k=1,2,\dots$	$1/p$	$(1-p)/p^2$
Uniform U(a,b)	$\frac{1}{b-a}$	$a\le x\le b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential Exp(λ)	$\frac{1}{\lambda}e^{-x/\lambda}$	$x\ge 0$	λ	λ^2
Gamma $\Gamma(a,b)$	$\frac{1}{b^a\Gamma(a)}x^{a-1}e^{-x/b}$	$x\ge 0$	ab	ab^2
Normal $\mathsf{N}(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$	$-\infty\le x\le\infty$	μ	σ^2
χ^2-distribution $\chi^2(n)$	$\frac{1}{2^{n/2}\Gamma(\frac{n}{2})}e^{-x/2}x^{n/2-1}$	$x\ge 0$	n	$2n$
t-distribution t(ν)	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	$-\infty\le x\le\infty$	0	$\frac{\nu}{\nu-2}$ if $\nu>2$ ∞ if $1<\nu\le 2$
F-distribution F(n,m)	$\frac{\Gamma(\frac{n+m}{2})n^{\frac{n}{2}}m^{\frac{m}{2}}x^{\frac{n-2}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})(m+nx)^{\frac{n+m}{2}}}$	$x\ge 0$	$\frac{m}{m-2}$ if $m>2$	$\frac{2m^2(m+n-2)}{n(m-2)^2(m-4)}$ if $m>4$

Table 1: Common distributions, where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ and $\Gamma(\nu)$ is the Gamma function with $\Gamma(k)=(k-1)!$ for positive integers k.

Interval estimators

All confidence intervals below are two-sided with a confidence level of $100(1-\alpha)\%$

- μ where $X_i\sim\mathsf{N}(\mu,\sigma^2)$ distribution and σ is known: $I_\mu=\left(\bar{x}\pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$
- μ where $X_i\sim\mathsf{N}(\mu,\sigma^2)$ and σ is unknown: $I_\mu=\left(\bar{x}\pm t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right)$
- σ^2 where $X_i\sim\mathsf{N}(\mu,\sigma^2)$ and μ is unknown: $I_{\sigma^2}=\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}(n-1)},\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}(n-1)}\right)$
- $\mu_1-\mu_2$ where $X_i\sim\mathsf{N}(\mu_1,\sigma_1^2), i=1,\dots,n_1$ and $Y_i\sim\mathsf{N}(\mu_2,\sigma_2^2), i=1,\dots,n_2$

- where σ_1 and σ_2 are known: $I_{\mu_1-\mu_2} = \left(\bar{x} - \bar{y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$
- where $\sigma_1 = \sigma_2 = \sigma$ and σ is unknown: $I_{\mu_1-\mu_2} = \left(\bar{x} - \bar{y} \pm t_{\alpha/2}(n_1 + n_2 - 2)s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$, here is $s_p^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$ the pooled variance estimator.
- where $\sigma_1 \neq \sigma_2$ are unknown (approximative): $I_{\mu_1-\mu_2} = \left(\bar{x} - \bar{y} \pm t_{\alpha/2}(f)\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right)$, here $f = \frac{(s_1^2/n_1+s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1-1}+\frac{(s_2^2/n_2)^2}{n_2-1}}$

- σ_1^2/σ_2^2 where $X_i \sim \mathsf{N}(\mu_1, \sigma_1^2), i = 1, \dots, n_1$ and $Y_i \sim \mathsf{N}(\mu_2, \sigma_2^2), i = 1, \dots, n_2$
  μ_1 and μ_2 are unknown $I_{\sigma_1^2/\sigma_2^2} = \left(\frac{s_1^2/s_2^2}{F_{\alpha/2}(n_1-1,n_2-1)}, \frac{s_1^2/s_2^2}{F_{1-\alpha/2}(n_1-1,n_2-1)}\right)$
- Δ where $Z_i = X_i - Y_i \sim \mathsf{N}(\Delta, \sigma^2), i = 1, \dots, n$ and σ is unknown (paired sample): $I_\Delta = \left(\bar{z} \pm t_{\alpha/2}(n-1)\frac{s_z}{\sqrt{n}}\right)$
- p where $X \sim \text{Bin}(n, p)$ (approximative with at least $np(1-p) \geq 10$): $I_p = \left(p^* \pm z_{\alpha/2}\sqrt{\frac{p^*(1-p^*)}{n}}\right)$, with $p^* = \frac{x}{n}$
- $p_1 - p_2$ where $X_1 \sim \text{Bin}(n_1, p_1)$ distribution $X_2 \sim \text{Bin}(n_2, p_2)$ (approximative with at least $n_i p_i(1-p_i) \geq 10$):
  $I_{p_1-p_2} = \left(p_1^* - p_2^* \pm z_{\alpha/2}\sqrt{\frac{p_1^*(1-p_1^*)}{n_1} + \frac{p_2^*(1-p_2^*)}{n_2}}\right)$, where $p_i^* = \frac{x_i}{n_i}$
- μ where $X \sim \text{Po}(\mu)$ (approximative with at least $\mu \geq 15$): $I_\mu = \left(x \pm z_{\alpha/2}\sqrt{x}\right)$

Simple linear regression

Model where $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, i = 1, \dots, n$, here $\varepsilon_i \sim \mathsf{N}(0, \sigma^2)$ are independent.

- Least-squares estimator

$$\beta_1^* = \frac{S_{xy}}{S_{xx}} \sim \mathsf{N}(\beta_1, \frac{\sigma^2}{S_{xx}}), \qquad \beta_0^* = \bar{y} - \beta_1^*\bar{x} \sim \mathsf{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$$

$$s^2 = \frac{Q_0}{n-2} \text{ with } Q_0 = \sum_{i=1}^n (y_i - \beta_0^* - \beta_1^* x_i)^2 = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

$$S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2, S_{yy} = \sum_{i=1}^n (y_i - \bar{y})^2, S_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

- Two-sided confidence interval for $\mu_Y(x_0) = \beta_0 + \beta_1 x_0$: $I_{\mu_Y(x_0)} = \left(\beta_0^* + \beta_1^* x_0 \pm t_{\alpha/2}(n-2)s\sqrt{\frac{1}{n} + \frac{(x_0-\bar{x})^2}{S_{xx}}}\right)$
- Two-sided prediction interval for $Y(x_0)$: $I_{Y(x_0)} = \left(\beta_0^* + \beta_1^* x_0 \pm t_{\alpha/2}(n-2)s\sqrt{1 + \frac{1}{n} + \frac{(x_0-\bar{x})^2}{S_{xx}}}\right)$

Multiple linear regression

Model is $y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi} + \varepsilon_i, i = 1, \dots, n$, here $\varepsilon_i \sim \mathsf{N}(0, \sigma^2)$ are independent.

- The model can also be written in matrix form as $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.
- Least-squares estimator

$$\boldsymbol{\beta}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \sim \mathsf{N}(\boldsymbol{\beta}, \sigma^2(\mathbf{X}^T\mathbf{X})^{-1})$$

$$s^2 = \frac{Q_0}{n-(p+1)} \text{ where } Q_0 = \sum_{i=1}^n (y_i - \beta_0^* - \beta_1^* x_{1i} - \cdots - \beta_p^* x_{pi})^2 = \mathbf{Y}^T\mathbf{Y} - \boldsymbol{\beta}^{*T}\mathbf{X}^T\mathbf{Y}$$

- Two-sided confidence interval for β_i: $I_{\beta_i} = \left(\beta_i^* \pm t_{\alpha/2}(n-p-1)s\sqrt{((\mathbf{X}^T\mathbf{X})^{-1})_{i+1,i+1}}\right)$
- Two-sided confidence interval for $\mu_Y(\mathbf{x}_0) = \beta_0 + \beta_1 x_{10} + \cdots + \beta_p x_{p0}$:
  $I_{\mu_Y(\mathbf{x}_0)} = \left(\mu_Y^*(\mathbf{x}_0) \pm t_{\alpha/2}(n-p-1)s\sqrt{\mathbf{x}_0^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_0}\right)$

Factorial experiments

A 2^k-factorial experiment analyses the influence of k factors: A, B, C, ... Each factor can take two levels. We have n replicates: each factor combination is measured n times. For example a 2^3-experiment is modelled by

$$Y_{ijkl} = \mu + Ax_i + Bx_j + Cx_k + ABx_ix_j + ACx_ix_k + BCx_jx_k + ABCx_ix_jx_k + \varepsilon_{ijkl}$$

where $x_i = -1$ if factor A has a low level $x_i = 1$ if factor A has a high level, and x_j and x_k are defined accordingly for factor B and C. The various effects can be estimated using the following symbolic table:

Experiment	Mean	μ	A	B	C	AB	AC	BC	ABC
(1)	$\bar{y}_{111}$	+	-	-	-	+	+	+	-
a	$\bar{y}_{211}$	+	+	-	-	-	-	+	+
b	$\bar{y}_{121}$	+	-	+	-	-	+	-	+
ab	$\bar{y}_{221}$	+	+	+	-	+	-	-	-
c	$\bar{y}_{112}$	+	-	-	+	+	-	-	+
ac	$\bar{y}_{212}$	+	+	-	+	-	+	-	-
bc	$\bar{y}_{122}$	+	-	+	+	-	-	+	-
abc	$\bar{y}_{222}$	+	+	+	+	+	+	+	+

For a given effect θ the standard error for the estimate is given by $SE(\hat{\theta}) = s/\sqrt{2^k n}$, where s^2 is the pooled variance estimate from the various experiments. The variance estimate has $2^k(n-1)$ degrees of freedom. If it is assumed that θ is zero, $2^k n\hat{\theta}^2$ is an estimator of the variance, which is useful if the experiment does not have any repetitions.

Analysis of variance (ANOVA)

A one-way ANOVA experiment is a model of the form

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \text{ where } \varepsilon_{ij} \sim \mathsf{N}(0,\sigma^2), i = 1,\ldots,a, j = 1,\ldots,n_i$$

and the following ANOVA table can be used to test if all $\alpha_i = 0$:

Variation	Sum of squares (SS)	Degrees of freedom	Mean square	Test statistic
Faktor A	$SS_A = \sum_{ij}(\bar{y}_i - \bar{y})^2$	$f_A = a-1$	$MS_A = SS_A/f_A$	MS_A/MS_E
Residual	$SS_E = \sum_{ij}(y_{ij} - \bar{y}_i)^2$	$f_E = \sum_i n_i - a$	$MS_E = SS_E/f_E$	
Total	$SS_{Tot} = SS_E + SS_A$	$f = \sum_i n_i - 1$		

A two-way ANOVA experiment is a model of the form

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \text{ where } \varepsilon_{ijk} \sim \mathsf{N}(0,\sigma^2), i = 1,\ldots,a, j = 1,\ldots,b, k = 1,\ldots,n$$

and the following ANOVA table can be used to test if all $\alpha_i = 0$, if all $\beta_i = 0$, or if all $(\alpha\beta)_{ij} = 0$:

Variation	Sum of squares (SS)	Degrees of freedom	Mean square	Test statistic
Factor A	$SS_A = \sum_{ijk}(\bar{y}_{i\cdot} - \bar{y})^2$	$f_A = a-1$	$MS_A = SS_A/f_A$	MS_A/MS_E
Factor B	$SS_B = \sum_{ijk}(\bar{y}_{\cdot j} - \bar{y})^2$	$f_B = b-1$	$MS_B = SS_B/f_B$	MS_B/MS_E
Factor AB	$SS_{AB} = SS_{Tot} - SS_E - SS_A - SS_B$	$f_{AB} = f_A f_B$	$MS_{AB} = SS_{AB}/f_{AB}$	MS_{AB}/MS_E
Residual	$SS_E = \sum_{ijk}(y_{ijk} - \bar{y}_{ij})^2$	$f_E = ab(n-1)$	$MS_E = SS_E/f_E$	
Total	$SS_{Tot} = \sum_{ijk}(y_{ijk} - \bar{y})^2$	$abn - 1$		

Test statistic for effect θ is a realisation of $F(f_\theta, f_E)$.

Sannolikhetsteorins grunder

- Följande gäller för sannolikheter
  - $0 \leq \mathsf{P}(A) \leq 1$
  - $\mathsf{P}(\Omega) = 1$
  - $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$ om händelserna A och B är oförenliga (disjunkta)
- Additionssatsen för två händelser: $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$
- Betingad sannolikhet: $\mathsf{P}(B|A) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(A)}$
- Bayes sats: $\mathsf{P}(A|B) = \frac{\mathsf{P}(B|A)\mathsf{P}(A)}{\mathsf{P}(B)}$
- Satsen om total sannolikhet: $\mathsf{P}(A) = \sum_{i=1}^{n} \mathsf{P}(A|B_i)\mathsf{P}(B_i)$, där händelserna $B_1, \ldots, B_n$ är parvis oförenliga händelser och $\bigcup_{i=1}^{n} B_i = \Omega$
- A och B är oberoende $\Leftrightarrow \mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B)$

Endimensionella stokastiska variabler

- Fördelningsfunktionen för X: $F_X(x) = \mathsf{P}(X \leq x)$
- Sannolikhetsfunktionen för en diskret s.v. X: $f_X(k) = \mathsf{P}(X = k)$
- Täthetsfunktionen för en kontinuerlig s.v. X: $f_X(x) = \frac{dF_X(x)}{dx}$
- $\mathsf{P}(a < X \leq b) = F_X(b) - F_X(a) = \begin{cases} \sum_{k=a+1}^{b} f_X(k) & \text{(diskret s.v. och } a \text{ och } b \text{ heltal)} \\ \int_a^b f_X(x)dx & \text{(kontinuerlig s.v.)} \end{cases}$

Flerdimensionella stokastiska variabler

- Simultan fördelningsfunktion:

$$F_{X,Y}(x,y) = \mathsf{P}(X \leq x, Y \leq y) = \begin{cases} \sum_{i \leq x, j \leq y} f_{X,Y}(i,j), & \text{(diskret s.v.)} \\ \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(t,u)dtdu, & \text{(kontinuerlig s.v.)} \end{cases}$$

- Marginell täthetsfunktion:

$$f_X(x) = \begin{cases} \sum_{y=-\infty}^{\infty} f_{X,Y}(x,y) & \text{(diskret s.v.)} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy & \text{(kontinuerlig s.v.)} \end{cases}$$

- Betingad fördelning: $f_{X|y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
- $\mathsf{P}(a \leq X \leq b \text{ och } c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)dxdy.$
- X och Y är oberoende om $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ för alla x och y.

Väntevärden

- Väntevärdet av $g(X,Y)$:

$$\mathsf{E}(g(X,Y)) = \begin{cases} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g(i,j)f_{X,Y}(i,j), & \text{(diskret s.v.)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy, & \text{(kontinuerlig s.v.)} \end{cases}$$

- Varians: $\mathsf{V}(X) = \mathsf{E}((X - \mathsf{E}(X))^2) = \mathsf{E}(X^2) - \mathsf{E}(X)^2$
- Standardavvikelse: $\sqrt{V(X)}$
- Kovarians: $\mathsf{C}(X,Y) = \mathsf{E}((X - \mathsf{E}(X))(Y - \mathsf{E}(Y))) = \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y)$

- Väntevärde av linjärkombintaion: $\mathsf{E}(\sum_{i=1}^n a_i X_i + b) = \sum_{i=1}^n a_i \mathsf{E}(X_i) + b$
- Varians av linjärkombination: $\mathsf{V}(\sum_{i=1}^n a_i X_i + b) = \sum_{i=1}^n a_i^2 \mathsf{V}(X_i) + 2\sum_{i=1}^n \sum_{j=i+1}^n a_i a_j \mathsf{C}(X_i, X_j)$
- Om $X_1, \ldots, X_n$ är oberoende så är de okorrelerade, dvs $\mathsf{C}(X_i, X_j) = 0, i \neq j$
- Korrelationskoefficient: $\rho(X,Y) = \frac{\mathsf{C}(X,Y)}{\sqrt{\mathsf{V}(X)\mathsf{V}(Y)}}$

Fördelning	$f(x)$		väntevärde	varians
Binomial Bin(n,p)	$\binom{n}{k} p^k (1-p)^{n-k}$	$k = 0, \ldots, n$	np	np(1-p)
Negativ binomial nBin(r,p)	$\binom{k-1}{r-1} p^r (1-p)^{k-r}$	$k = r, r+1 \ldots$	r/p	$r(1-p)/p^2$
Hypergeometrisk Hyp(N,n,r)	$\frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}$	$k = \max(0, n+r-N), \ldots, \min(n,r)$	nr/N	$\frac{nr(N-r)(N-n)}{N^2(N-1)}$
Poisson Po(μ)	$p(k) = \frac{\mu^k}{k!} e^{-\mu}$	$k = 0, 1, \ldots$	μ	μ
Geometrisk Ge(p)	$p(1-p)^{k-1}$	$k = 1, 2, \ldots$	$1/p$	$(1-p)/p^2$
Likformig U(a,b)	$\frac{1}{b-a}$	$a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential Exp(λ)	$\frac{1}{\lambda} e^{-x/\lambda}$	$x \geq 0$	λ	λ^2
Gamma $\Gamma(a,b)$	$\frac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b}$	$x \geq 0$	ab	ab^2
Normal $\mathsf{N}(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$	$-\infty \leq x \leq \infty$	μ	σ^2
χ^2-fördelning $\chi^2(n)$	$\frac{1}{2^{n/2}\Gamma(\frac{n}{2})} e^{-x/2} x^{n/2-1}$	$x \geq 0$	n	$2n$
t-fördelning t(ν)	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	$-\infty \leq x \leq \infty$	0	$\frac{\nu}{\nu-2}$ om $\nu > 2$ ∞ om $1 < \nu \leq 2$
F-fördelning F(n,m)	$\frac{\Gamma(\frac{n+m}{2}) n^{\frac{n}{2}} m^{\frac{m}{2}} x^{\frac{n-2}{2}}}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})(m+nx)^{\frac{n+m}{2}}}$	$x \geq 0$	$\frac{m}{m-2}$ om $m > 2$	$\frac{2m^2(m+n-2)}{n(m-2)^2(m-4)}$ om $m > 4$

Table 2: Vanliga fördelningar, där $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ och $\Gamma(\nu)$ är gammafunktionen som uppfyller $\Gamma(k) = (k-1)!$ för positiva heltal k.

Egenskaper hos vanliga fördelningar

- $X \sim$ Bin(n_1, p), $Y \sim$ Bin(n_2, p) samt oberoende $\Rightarrow X + Y \sim$Bin$(n_1 + n_2, p)$
- $X \sim$Po(μ_1), $Y \sim$Po(μ_2) samt oberoende $\Rightarrow X + Y \sim$Po$(\mu_1 + \mu_2)$

- $X \sim \mathsf{N}(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim \mathsf{N}(0,1)$
- $X \sim \mathsf{N}(\mu, \sigma^2) \Rightarrow F_X(x) = \Phi(\frac{x-\mu}{\sigma})$ där $\Phi(\cdot)$ ges av tabell.
- $X_i \sim \mathsf{N}(\mu_i, \sigma_i^2), i = 1, \ldots, n$ oberoende $\Rightarrow \sum_{i=1}^n a_i X_i \sim \mathsf{N}(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2)$
- $X \sim \mathsf{N}(0,1), Y \sim \chi^2(\nu)$ samt oberoende $\Rightarrow \frac{X}{\sqrt{Y/\nu}} \sim t(\nu)$
- $X_1, \ldots, X_n$ oberoende och $\mathsf{N}(0,1) \Rightarrow \sum_{i=1}^n X_i^2 \sim \chi^2(n)$
- $X_1, \ldots, X_n$ oberoende och $\mathsf{N}(\mu, \sigma^2) \Rightarrow \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2(n-1)$
- $X \sim \chi^2(n), Y \sim \chi^2(m)$ samt oberoende $\Rightarrow \frac{X/n}{Y/m} \sim F(n,m)$
- $F_{1-\alpha}(n,m) = 1/F_\alpha(m,n)$

Centrala gränsvärdessatsen

Om $X_1, \ldots, X_n$ är oberoende och likafördelade med $\mathsf{E}(X_i) = \mu_i$ och $\mathsf{V}(X_i) = \sigma^2$ så gäller att $\sum_{i=1}^n X_i$ är approximativt $\mathsf{N}(n\mu, n\sigma^2)$-fördelad om n är stort nog. Från bland annat detta följer följande approximationer

- $\text{Po}(\mu) \approx \mathsf{N}(\mu, \mu)$ om $\mu \geq 15$
- $\text{Bin}(n,p) \approx \mathsf{N}(np, np(1-p))$ om $np(1-p) \geq 10$
- $\text{Bin}(n,p) \approx \text{Po}(np)$ om $p \leq 0.1$ och $n \geq 10$

Statistik och punktskattningar

Beskrivning av data:

- Stickprovsmedelvärde: $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$
- Stickprovsvarians: $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 - n \cdot \bar{x}^2\right]$
- Stickprovskovarians: $c_{xy} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$
- Korrelationskoefficient: $r_{xy} = \frac{c_{xy}}{s_x s_y}$

Låt $x_1, \ldots, x_n$ vara observationer av oberoende och likafördelade s.v. $X_1, \ldots, X_n$ med väntevärde μ och varians σ^2, då är stickprovsmedelvärdet en väntevärdesriktig skattning av μ och stickprovsvariansen en väntevärdesriktig skattning av σ^2.

Intervallskattningar

Samtliga intervall nedan är tvåsidiga med $100(1-\alpha)\%$ konfidensgrad

- μ då $X_i \sim \mathsf{N}(\mu, \sigma^2)$ och σ är känd: $I_\mu = \left(\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$
- μ då $X_i \sim \mathsf{N}(\mu, \sigma^2)$ och σ är okänd: $I_\mu = \left(\bar{x} \pm t_{\alpha/2}(n-1) \frac{s}{\sqrt{n}}\right)$
- σ^2 då $X_i \sim \mathsf{N}(\mu, \sigma^2)$ och μ är okänd: $I_{\sigma^2} = \left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}(n-1)}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}(n-1)}\right)$
- $\mu_1 - \mu_2$ då $X_i \sim \mathsf{N}(\mu_1, \sigma_1^2), i = 1, \ldots, n_1$ och $Y_i \sim \mathsf{N}(\mu_2, \sigma_2^2), i = 1, \ldots, n_2$
  - då σ_1 och σ_2 är kända: $I_{\mu_1-\mu_2} = \left(\bar{x} - \bar{y} \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$
  - då $\sigma_1 = \sigma_2 = \sigma$ där σ är okänd: $I_{\mu_1-\mu_2} = \left(\bar{x} - \bar{y} \pm t_{\alpha/2}(n_1+n_2-2) s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$,
    här är $s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$ den poolade variansskattningen.

* $*$ då $\sigma_1 \neq \sigma_2$ är okända (approximativt): $I_{\mu_1-\mu_2} = \left(\bar{x} - \bar{y} \pm t_{\alpha/2}(f)\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right)$, där $f = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$

- σ_1^2/σ_2^2 då $X_i \sim \mathsf{N}(\mu_1, \sigma_1^2), i = 1, \ldots, n_1$ och $Y_i \sim \mathsf{N}(\mu_2, \sigma_2^2), i = 1, \ldots, n_2$
  μ_1 och μ_2 okända $I_{\sigma_1^2/\sigma_2^2} = \left(\frac{s_1^2/s_2^2}{F_{\alpha/2}(n_1-1, n_2-1)}, \frac{s_1^2/s_2^2}{F_{1-\alpha/2}(n_1-1, n_2-1)}\right)$

- Δ då $Z_i = X_i - Y_i \sim \mathsf{N}(\Delta, \sigma^2), i = 1, \ldots, n$ där σ okänd (stickprov i par): $I_\Delta = \left(\bar{z} \pm t_{\alpha/2}(n-1)\frac{s_z}{\sqrt{n}}\right)$

- p då $X \sim \mathrm{Bin}(n, p)$ (approximativt då $np(1-p) \geq 10$): $I_p = \left(p^* \pm z_{\alpha/2}\sqrt{\frac{p^*(1-p^*)}{n}}\right)$, där $p^* = \frac{x}{n}$

- $p_1 - p_2$ då $X_1 \sim \mathrm{Bin}(n_1, p_1)$ och $X_2 \sim \mathrm{Bin}(n_2, p_2)$ (approximativt då $n_i p_i(1-p_i) \geq 10$):
  $I_{p_1-p_2} = \left(p_1^* - p_2^* \pm z_{\alpha/2}\sqrt{\frac{p_1^*(1-p_1^*)}{n_1} + \frac{p_2^*(1-p_2^*)}{n_2}}\right)$, där $p_i^* = \frac{x_i}{n_i}$

- μ då $X \sim \mathrm{Po}(\mu)$ (approximativt då $\mu \geq 15$): $I_\mu = \left(x \pm z_{\alpha/2}\sqrt{x}\right)$

Enkel linjär regression

Modellen är $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, i = 1, \ldots, n$ där $\varepsilon_i \sim \mathsf{N}(0, \sigma^2)$ är oberoende.

- Minsta-kvadratskattningar

$$\beta_1^* = \frac{S_{xy}}{S_{xx}} \sim \mathsf{N}(\beta_1, \frac{\sigma^2}{S_{xx}}), \qquad \beta_0^* = \bar{y} - \beta_1^*\bar{x} \sim \mathsf{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$$

$$s^2 = \frac{Q_0}{n-2} \text{ där } Q_0 = \sum_{i=1}^{n}(y_i - \beta_0^* - \beta_1^* x_i)^2 = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2, S_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2, S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

- Tvåsidigt konfidensintervall för $\mu_Y(x_0) = \beta_0 + \beta_1 x_0$: $I_{\mu_Y(x_0)} = \left(\beta_0^* + \beta_1^* x_0 \pm t_{\alpha/2}(n-2)s\sqrt{\frac{1}{n} + \frac{(x_0-\bar{x})^2}{S_{xx}}}\right)$

- Tvåsidigt prediktionsintervall för $Y(x_0)$: $I_{Y(x_0)} = \left(\beta_0^* + \beta_1^* x_0 \pm t_{\alpha/2}(n-2)s\sqrt{1 + \frac{1}{n} + \frac{(x_0-\bar{x})^2}{S_{xx}}}\right)$

Multipel linjär regression

Modellen är $y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi} + \varepsilon_i, i = 1, \ldots, n$ där $\varepsilon_i \sim \mathsf{N}(0, \sigma^2)$ är oberoende.

- Modellen kan också skrivas på matrisform som $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.
- Minsta-kvadratskattningar

$$\boldsymbol{\beta}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \sim \mathsf{N}(\boldsymbol{\beta}, \sigma^2(\mathbf{X}^T\mathbf{X})^{-1})$$

$$s^2 = \frac{Q_0}{n-(p+1)} \text{ där } Q_0 = \sum_{i=1}^{n}(y_i - \beta_0^* - \beta_1^* x_{1i} - \cdots - \beta_p^* x_{pi})^2 = \mathbf{Y}^T\mathbf{Y} - \boldsymbol{\beta}^{*T}\mathbf{X}^T\mathbf{Y}$$

- Tvåsidigt konfidensintervall för β_i: $I_{\beta_i} = \left(\beta_i^* \pm t_{\alpha/2}(n-p-1)s\sqrt{((\mathbf{X}^T\mathbf{X})^{-1})_{i+1,i+1}}\right)$

- Tvåsidigt konfidensintervall för $\mu_Y(\mathbf{x}_0) = \beta_0 + \beta_1 x_{10} + \cdots + \beta_p x_{p0}$:
  $I_{\mu_Y(\mathbf{x}_0)} = \left(\mu_Y^*(\mathbf{x}_0) \pm t_{\alpha/2}(n-p-1)s\sqrt{\mathbf{x}_0^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_0}\right)$

Faktorförsök

I ett 2^k-faktorförsök undersöks k faktorer: A,B, C, ... Varje faktor kan anta två nivåer. Vid n replikat mäts varje faktorkombination n gånger. Till exempel för ett 2^3-försök är modellen

$$Y_{ijkl} = \mu + Ax_i + Bx_j + Cx_k + ABx_ix_j + ACx_ix_k + BCx_jx_k + ABCx_ix_jx_k + \varepsilon_{ijkl}$$

där $x_i = -1$ om faktor A har låg nivå och $x_i = 1$ om faktor A har hög nivå, och x_j och x_k är definierade på motsvarande sätt för faktor B och C. De olika effekterna kan skattas med hjälp av följande teckentabell

Försök	Medel	μ	A	B	C	AB	AC	BC	ABC
(1)	$\bar{y}_{111}$	+	-	-	-	+	+	+	-
a	$\bar{y}_{211}$	+	+	-	-	-	-	+	+
b	$\bar{y}_{121}$	+	-	+	-	-	+	-	+
ab	$\bar{y}_{221}$	+	+	+	-	+	-	-	-
c	$\bar{y}_{112}$	+	-	-	+	+	-	-	+
ac	$\bar{y}_{212}$	+	+	-	+	-	+	-	-
bc	$\bar{y}_{122}$	+	-	+	+	-	-	+	-
abc	$\bar{y}_{222}$	+	+	+	+	+	+	+	+

För en given effekt θ ges standardfelet för effektskattning av $d(\hat{\theta}) = s/\sqrt{2^k n}$, där s^2 är den poolade variansskattningen från de olika försöken. Variansskattningen har $2^k(n-1)$ frihetsgrader. Om θ kan antas vara noll är $2^k n\hat{\theta}^2$ en skattning av variansen, vilket är användbart om försöket saknar replikat.

Variansanalys

Ett försök med ensidig indelning har modell

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij},\ \text{där}\ , \varepsilon_{ij} \sim \mathsf{N}(0, \sigma^2), i = 1, \ldots, a, j = 1, \ldots, n_i$$

och följande variansanalystabell kan användas för att testa om alla $\alpha_i = 0$:

Variation	Kvadratsumma	Frihetsgrader	Medelkvadratsumma	Teststorhet
Faktor A	$SS_A = \sum_{ij}(\bar{y}_i - \bar{y})^2$	$f_A = a - 1$	$MS_A = SS_A/f_A$	MS_A/MS_E
Residual	$SS_E = \sum_{ij}(y_{ij} - \bar{y}_i)^2$	$f_E = \sum_i n_i - a$	$MS_E = SS_E/f_E$	
Total	$SS_{Tot} = SS_E + SS_A$	$f = \sum_i n_i - 1$		

Ett försök med tvåsidig indelning har modell

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},\ \text{där}\ , \varepsilon_{ijk} \sim \mathsf{N}(0, \sigma^2), i = 1, \ldots, a, j = 1, \ldots, b, k = 1, \ldots, n$$

och följande variansanalystabell kan användas för att testa om alla $\alpha_i = 0$, om alla $\beta_i = 0$, eller om alla $(\alpha\beta)_{ij} = 0$:

Variation	Kvadratsumma (SS)	Frihetsgrader	Medelkvadrat	Teststorhet
Faktor A	$SS_A = \sum_{ijk}(\bar{y}_{i\cdot} - \bar{y})^2$	$f_A = a - 1$	$MS_A = SS_A/f_A$	MS_A/MS_E
Faktor B	$SS_B = \sum_{ijk}(\bar{y}_{\cdot j} - \bar{y})^2$	$f_B = b - 1$	$MS_B = SS_B/f_B$	MS_B/MS_E
Faktor AB	$SS_{AB} = SS_{Tot} - SS_E - SS_A - SS_B$	$f_{AB} = f_A f_B$	$MS_{AB} = SS_{AB}/f_{AB}$	MS_{AB}/MS_E
Residual	$SS_E = \sum_{ijk}(y_{ijk} - \bar{y}_{ij})^2$	$f_E = ab(n-1)$	$MS_E = SS_E/f_E$	
Total	$SS_{Tot} = \sum_{ijk}(y_{ijk} - \bar{y})^2$	$abn - 1$		

Teststorheten för effekt θ en observation av $F(f_\theta, f_E)$.

Table 1: Normal distribution / Normalfördelningen

Table gives $\Phi(x) = \mathsf{P}(X \leq x)$ for $X \sim \mathsf{N}(0, 1)$. For negative values use that $\Phi(-x) = 1 - \Phi(x)$.
Tabellen visar $\Phi(x) = \mathsf{P}(X \leq x)$ där $X \sim \mathsf{N}(0, 1)$. För negativa värden, utnyttja att $\Phi(-x) = 1 - \Phi(x)$.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0 :	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1 :	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2 :	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3 :	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4 :	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5 :	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6 :	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7 :	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8 :	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9 :	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0 :	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1 :	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2 :	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3 :	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4 :	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5 :	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6 :	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7 :	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8 :	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9 :	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0 :	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1 :	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2 :	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3 :	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4 :	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5 :	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952

Table 2: Quantiles of the normal distribution / Normalfördelningens kvantiler

Table gives $\mathsf{P}(X > \lambda_\alpha) = \alpha$ for $X \sim \mathsf{N}(0, 1)$
Tabellen visar $\mathsf{P}(X > \lambda_\alpha) = \alpha$ där $X \sim \mathsf{N}(0, 1)$

α	.1	.05	.025	.01	.005	.001	.0005	.0001	.00005	.00001
λ_α	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905	3.7190	3.8906	4.2649

Table 3: Quantiles of the t-distribution / t-fördelningens kvantiler

Table gives $\mathsf{P}(X > t_\alpha(f)) = \alpha$ for $X \sim t(f)$
Tabellen visar $\mathsf{P}(X > t_\alpha(f)) = \alpha$ där $X \sim t(f)$

α	.1	.05	.025	.01	.005	.001	.0005
$t_\alpha(1)$	3.0777	6.3138	12.706	31.820	63.657	318.31	636.62
$t_\alpha(2)$	1.8856	2.9200	4.3027	6.9646	9.9248	22.327	31.599
$t_\alpha(3)$	1.6377	2.3534	3.1824	4.5407	5.8409	10.215	12.924
$t_\alpha(4)$	1.5332	2.1318	2.7764	3.7469	4.6041	7.1732	8.6103
$t_\alpha(5)$	1.4759	2.0150	2.5706	3.3649	4.0321	5.8934	6.8688
$t_\alpha(6)$	1.4398	1.9432	2.4469	3.1427	3.7074	5.2076	5.9588
$t_\alpha(7)$	1.4149	1.8946	2.3646	2.9980	3.4995	4.7853	5.4079
$t_\alpha(8)$	1.3968	1.8595	2.3060	2.8965	3.3554	4.5008	5.0413
$t_\alpha(9)$	1.3830	1.8331	2.2622	2.8214	3.2498	4.2968	4.7809
$t_\alpha(10)$	1.3722	1.8125	2.2281	2.7638	3.1693	4.1437	4.5869
$t_\alpha(11)$	1.3634	1.7959	2.2010	2.7181	3.1058	4.0247	4.4370
$t_\alpha(12)$	1.3562	1.7823	2.1788	2.6810	3.0545	3.9296	4.3178
$t_\alpha(13)$	1.3502	1.7709	2.1604	2.6503	3.0123	3.8520	4.2208
$t_\alpha(14)$	1.3450	1.7613	2.1448	2.6245	2.9768	3.7874	4.1405
$t_\alpha(15)$	1.3406	1.7531	2.1314	2.6025	2.9467	3.7328	4.0728
$t_\alpha(16)$	1.3368	1.7459	2.1199	2.5835	2.9208	3.6862	4.0150
$t_\alpha(17)$	1.3334	1.7396	2.1098	2.5669	2.8982	3.6458	3.9651
$t_\alpha(18)$	1.3304	1.7341	2.1009	2.5524	2.8784	3.6105	3.9216
$t_\alpha(19)$	1.3277	1.7291	2.0930	2.5395	2.8609	3.5794	3.8834
$t_\alpha(20)$	1.3253	1.7247	2.0860	2.5280	2.8453	3.5518	3.8495
$t_\alpha(21)$	1.3232	1.7207	2.0796	2.5176	2.8314	3.5272	3.8193
$t_\alpha(22)$	1.3212	1.7171	2.0739	2.5083	2.8188	3.5050	3.7921
$t_\alpha(23)$	1.3195	1.7139	2.0687	2.4999	2.8073	3.4850	3.7676
$t_\alpha(24)$	1.3178	1.7109	2.0639	2.4922	2.7969	3.4668	3.7454
$t_\alpha(25)$	1.3163	1.7081	2.0595	2.4851	2.7874	3.4502	3.7251
$t_\alpha(26)$	1.3150	1.7056	2.0555	2.4786	2.7787	3.4350	3.7066
$t_\alpha(27)$	1.3137	1.7033	2.0518	2.4727	2.7707	3.4210	3.6896
$t_\alpha(28)$	1.3125	1.7011	2.0484	2.4671	2.7633	3.4082	3.6739
$t_\alpha(29)$	1.3114	1.6991	2.0452	2.4620	2.7564	3.3962	3.6594
$t_\alpha(30)$	1.3104	1.6973	2.0423	2.4573	2.7500	3.3852	3.6460
$t_\alpha(40)$	1.3031	1.6839	2.0211	2.4233	2.7045	3.3069	3.5510
$t_\alpha(60)$	1.2958	1.6706	2.0003	2.3901	2.6603	3.2317	3.4602
$t_\alpha(120)$	1.2886	1.6577	1.9799	2.3578	2.6174	3.1595	3.3735
$t_\alpha(\infty)$	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

Table 4: Quantiles of the χ^2-distribution / χ^2-fördelningens kvantiler

Tabellen visar $\mathsf{P}(X > \chi^2_\alpha(f)) = \alpha$ där $X \sim \chi^2(f)$
Table gives $\mathsf{P}(X > \chi^2_\alpha(f)) = \alpha$ for $X \sim \chi^2(f)$

α	.9995	.999	.995	.99	.975	.95	.05	.025	.01	.005	.001	.0005
$\chi^2_\alpha(1)$	0.00	0.00	0.00	0.00	0.00	0.00	3.84	5.02	6.63	7.88	10.8	12.1
$\chi^2_\alpha(2)$	0.00	0.00	0.01	0.02	0.05	0.10	5.99	7.38	9.21	10.6	13.8	15.2
$\chi^2_\alpha(3)$	0.02	0.02	0.07	0.12	0.22	0.35	7.81	9.35	11.3	12.8	16.3	17.7
$\chi^2_\alpha(4)$	0.06	0.09	0.21	0.30	0.48	0.71	9.49	11.1	13.3	14.9	18.5	20.0
$\chi^2_\alpha(5)$	0.16	0.21	0.41	0.55	0.83	1.15	11.1	12.8	15.1	16.7	20.5	22.1
$\chi^2_\alpha(6)$	0.30	0.38	0.68	0.87	1.24	1.64	12.6	14.4	16.8	18.5	22.5	24.1
$\chi^2_\alpha(7)$	0.48	0.60	0.99	1.24	1.69	2.17	14.1	16.0	18.5	20.3	24.3	26.0
$\chi^2_\alpha(8)$	0.71	0.86	1.34	1.65	2.18	2.73	15.5	17.5	20.1	22.0	26.1	27.9
$\chi^2_\alpha(9)$	0.97	1.15	1.73	2.09	2.70	3.33	16.9	19.0	21.7	23.6	27.9	29.7
$\chi^2_\alpha(10)$	1.26	1.48	2.16	2.56	3.25	3.94	18.3	20.5	23.2	25.2	29.6	31.4
$\chi^2_\alpha(11)$	1.59	1.83	2.60	3.05	3.82	4.57	19.7	21.9	24.7	26.8	31.3	33.1
$\chi^2_\alpha(12)$	1.93	2.21	3.07	3.57	4.40	5.23	21.0	23.3	26.2	28.3	32.9	34.8
$\chi^2_\alpha(13)$	2.31	2.62	3.57	4.11	5.01	5.89	22.4	24.7	27.7	29.8	34.5	36.5
$\chi^2_\alpha(14)$	2.70	3.04	4.07	4.66	5.63	6.57	23.7	26.1	29.1	31.3	36.1	38.1
$\chi^2_\alpha(15)$	3.11	3.48	4.60	5.23	6.26	7.26	25.0	27.5	30.6	32.8	37.7	39.7
$\chi^2_\alpha(16)$	3.54	3.94	5.14	5.81	6.91	7.96	26.3	28.8	32.0	34.3	39.3	41.3
$\chi^2_\alpha(17)$	3.98	4.42	5.70	6.41	7.56	8.67	27.6	30.2	33.4	35.7	40.8	42.9
$\chi^2_\alpha(18)$	4.44	4.90	6.26	7.01	8.23	9.39	28.9	31.5	34.8	37.2	42.3	44.4
$\chi^2_\alpha(19)$	4.91	5.41	6.84	7.63	8.91	10.1	30.1	32.9	36.2	38.6	43.8	46.0
$\chi^2_\alpha(20)$	5.40	5.92	7.43	8.26	9.59	10.9	31.4	34.2	37.6	40.0	45.3	47.5
$\chi^2_\alpha(21)$	5.90	6.45	8.03	8.90	10.3	11.6	32.7	35.5	38.9	41.4	46.8	49.0
$\chi^2_\alpha(22)$	6.40	6.98	8.64	9.54	11.0	12.3	33.9	36.8	40.3	42.8	48.3	50.5
$\chi^2_\alpha(23)$	6.92	7.53	9.26	10.2	11.7	13.1	35.2	38.1	41.6	44.2	49.7	52.0
$\chi^2_\alpha(24)$	7.45	8.08	9.89	10.9	12.4	13.8	36.4	39.4	43.0	45.6	51.2	53.5
$\chi^2_\alpha(25)$	7.99	8.65	10.5	11.5	13.1	14.6	37.7	40.6	44.3	46.9	52.6	54.9
$\chi^2_\alpha(26)$	8.54	9.22	11.2	12.2	13.8	15.4	38.9	41.9	45.6	48.3	54.1	56.4
$\chi^2_\alpha(27)$	9.09	9.80	11.8	12.9	14.6	16.2	40.1	43.2	47.0	49.6	55.5	57.9
$\chi^2_\alpha(28)$	9.66	10.4	12.5	13.6	15.3	16.9	41.3	44.5	48.3	51.0	56.9	59.3
$\chi^2_\alpha(29)$	10.2	11.0	13.1	14.3	16.0	17.7	42.6	45.7	49.6	52.3	58.3	60.7
$\chi^2_\alpha(30)$	10.8	11.6	13.8	15.0	16.8	18.5	43.8	47.0	50.9	53.7	59.7	62.2
$\chi^2_\alpha(40)$	16.9	17.9	20.7	22.2	24.4	26.5	55.8	59.3	63.7	66.8	73.4	76.1
$\chi^2_\alpha(50)$	23.5	24.7	28.0	29.7	32.4	34.8	67.5	71.4	76.2	79.5	86.7	89.6
$\chi^2_\alpha(60)$	30.3	31.7	35.5	37.5	40.5	43.2	79.1	83.3	88.4	92.0	99.6	103
$\chi^2_\alpha(70)$	37.5	39.0	43.3	45.4	48.8	51.7	90.5	95.0	100	104	112	116
$\chi^2_\alpha(80)$	44.8	46.5	51.2	53.5	57.2	60.4	102	107	112	116	125	128
$\chi^2_\alpha(90)$	52.3	54.2	59.2	61.8	65.6	69.1	113	118	124	128	137	141
$\chi^2_\alpha(100)$	59.9	61.9	67.3	70.1	74.2	77.9	124	130	136	140	149	153

Table 5: Quantiles of the F-distribution / F-fördelningens kvantiler

Tables give the values $F_\alpha(\nu_1, \nu_2)$ such that $\mathsf{P}(X > F_\alpha(\nu_1, \nu_2)) = \alpha$ for $X \sim F(\nu_1, \nu_2)$. Tabellerna på följande sidor visar tal $F_\alpha(\nu_1, \nu_2)$ så att $\mathsf{P}(X > F_\alpha(\nu_1, \nu_2)) = \alpha$ där $X \sim F(\nu_1, \nu_2)$.

		ν_2											
ν_1	α	1	2	3	4	5	6	7	8	9	10	11	12
1	0.025	647	38.5	17.4	12.2	10.0	8.81	8.07	7.57	7.20	6.93	6.72	6.55
	0.050	161	18.5	10.1	7.70	6.60	5.98	5.59	5.31	5.11	4.96	4.84	4.74
	0.950	.006	.005	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004
	0.975	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
2	0.025	799	39	16.0	10.6	8.43	7.26	6.54	6.05	5.71	5.45	5.25	5.09
	0.050	199	19	9.55	6.94	5.78	5.14	4.73	4.45	4.25	4.10	3.98	3.88
	0.950	.054	.052	.052	.052	.051	.051	.051	.051	.051	.051	.051	.051
	0.975	.026	.025	.025	.025	.025	.025	.025	.025	.025	.025	.025	.025
3	0.025	864	39.1	15.4	9.97	7.76	6.59	5.89	5.41	5.07	4.82	4.63	4.47
	0.050	215	19.1	9.27	6.59	5.40	4.75	4.34	4.06	3.86	3.70	3.58	3.49
	0.950	.098	.104	.107	.109	.110	.111	.112	.113	.113	.113	.114	.114
	0.975	.057	.062	.064	.066	.067	.067	.068	.068	.069	.069	.069	.069
4	0.025	899	39.2	15.1	9.60	7.38	6.22	5.52	5.05	4.71	4.46	4.27	4.12
	0.050	224	19.2	9.11	6.38	5.19	4.53	4.12	3.83	3.63	3.47	3.35	3.25
	0.950	.129	.144	.151	.156	.159	.162	.164	.165	.166	.167	.168	.169
	0.975	.081	.093	.100	.104	.106	.108	.110	.111	.112	.113	.113	.114
5	0.025	921	39.3	14.8	9.36	7.14	5.98	5.28	4.81	4.48	4.23	4.04	3.89
	0.050	230	19.3	9.01	6.25	5.05	4.38	3.97	3.68	3.48	3.32	3.20	3.10
	0.950	.151	.172	.184	.192	.198	.202	.205	.207	.209	.211	.212	.213
	0.975	.099	.118	.128	.135	.139	.143	.145	.148	.149	.151	.152	.153
6	0.025	937	39.3	14.7	9.19	6.97	5.82	5.11	4.65	4.32	4.07	3.88	3.72
	0.050	234	19.3	8.94	6.16	4.95	4.28	3.86	3.58	3.37	3.21	3.09	2.99
	0.950	.167	.194	.210	.220	.227	.233	.237	.241	.244	.246	.248	0.25
	0.975	.113	.137	.151	.160	.167	.171	.175	.178	.181	.183	.184	.186
7	0.025	948	39.3	14.6	9.07	6.85	5.69	4.99	4.52	4.19	3.95	3.75	3.60
	0.050	236	19.3	8.88	6.09	4.87	4.20	3.78	3.5	3.29	3.13	3.01	2.91
	0.950	.178	.211	.230	.242	.251	.258	.264	.268	.272	.275	.277	.279
	0.975	.123	.152	.169	.181	.189	.195	.200	.204	.207	0.21	.212	.214
8	0.025	956	39.3	14.5	8.98	6.75	5.6	4.89	4.43	4.10	3.85	3.66	3.51
	0.050	238	19.3	8.84	6.04	4.81	4.14	3.72	3.43	3.23	3.07	2.94	2.84
	0.950	.188	.224	.245	.260	.271	.279	.285	.290	.295	.298	.301	.304
	0.975	.132	.165	.184	.197	.207	.215	.220	.225	.229	.232	.235	.238
9	0.025	963	39.3	14.4	8.90	6.68	5.52	4.82	4.35	4.02	3.77	3.58	3.43
	0.050	240	19.3	8.81	5.99	4.77	4.09	3.67	3.38	3.17	3.02	2.89	2.79
	0.950	.195	.234	.258	.275	.287	.296	.303	.309	.314	.318	.322	.325
	0.975	.138	.175	.196	.212	.223	.231	.238	.243	.248	.252	.255	.258
10	0.025	968	39.4	14.4	8.84	6.61	5.46	4.76	4.29	3.96	3.71	3.52	3.37
	0.050	241	19.4	8.78	5.96	4.73	4.06	3.63	3.34	3.13	2.97	2.85	2.75
	0.950	.201	.243	.269	.287	.300	.310	.318	.325	.331	.335	.339	.343
	0.975	.144	.183	.207	.223	.236	.245	.253	.259	.264	.269	.272	.276
11	0.025	973	39.4	14.3	8.79	6.56	5.41	4.70	4.24	3.91	3.66	3.47	3.32
	0.050	243	19.4	8.76	5.93	4.70	4.02	3.60	3.31	3.10	2.94	2.81	2.71
	0.950	.206	.251	.278	.297	.312	.323	.332	.339	.345	.350	.354	.358
	0.975	.148	.190	.216	.233	.247	.257	.266	.272	.278	.283	.287	.291
12	0.025	976	39.4	14.3	8.75	6.52	5.36	4.66	4.2	3.86	3.62	3.43	3.27
	0.050	243	19.4	8.74	5.91	4.67	4	3.57	3.28	3.07	2.91	2.78	2.68
	0.950	.210	.257	.286	.306	.322	.333	.343	.351	.357	.363	.368	.372
	0.975	.152	.196	.223	.242	.257	.268	.277	.284	.291	.296	.301	.305

ν_1	α	ν_2 1	2	3	4	5	6	7	8	9	10	11	12
13	0.025	979	39.4	14.3	8.71	6.48	5.32	4.62	4.16	3.83	3.58	3.39	3.23
	0.050	244	19.4	8.72	5.89	4.65	3.97	3.55	3.25	3.04	2.88	2.76	2.66
	0.950	.214	.262	.293	.314	.330	.343	.353	.361	.368	.374	.379	.384
	0.975	.155	.201	0.23	.250	.265	.277	.287	.295	.301	.307	.312	.317
14	0.025	982	39.4	14.2	8.68	6.45	5.29	4.59	4.13	3.79	3.55	3.35	3.20
	0.050	245	19.4	8.71	5.87	4.63	3.95	3.52	3.23	3.02	2.86	2.73	2.63
	0.950	.217	.267	.299	.321	.338	.351	.361	.370	.378	.384	.389	.394
	0.975	.158	.205	.235	.256	.273	.285	.295	.304	.311	.317	.323	.327
15	0.025	984	39.4	14.2	8.65	6.42	5.26	4.56	4.10	3.76	3.52	3.33	3.17
	0.050	245	19.4	8.70	5.85	4.61	3.93	3.51	3.21	3.00	2.84	2.71	2.61
	0.950	.220	.271	.304	.327	.344	.358	.369	.378	.386	.393	.398	.404
	0.975	.161	.209	.240	.262	.279	.292	.303	.312	.320	.326	.332	.337
20	0.025	993	39.4	14.1	8.56	6.32	5.16	4.46	3.99	3.66	3.41	3.22	3.07
	0.050	248	19.4	8.66	5.80	4.55	3.87	3.44	3.15	2.93	2.77	2.64	2.54
	0.950	.229	.286	.322	.348	.368	.384	.397	.408	.417	.425	.432	.439
	0.975	.170	.224	.259	.284	.304	.319	.332	.343	.352	.360	.367	.373
25	0.025	998	39.4	14.1	8.50	6.26	5.10	4.40	3.93	3.60	3.35	3.16	3.00
	0.050	249	19.4	8.63	5.76	4.52	3.83	3.40	3.10	2.89	2.73	2.60	2.49
	0.950	.235	.295	.334	.362	.384	.401	.415	.427	.438	.447	.455	.461
	0.975	.175	.233	.270	.298	.319	.336	.351	.363	.373	.382	.390	.397
30	0.025	1001	39.4	14.0	8.46	6.22	5.06	4.36	3.89	3.56	3.31	3.11	2.96
	0.050	250	19.4	8.61	5.74	4.49	3.80	3.37	3.07	2.86	2.7	2.57	2.46
	0.950	.239	.301	.342	.371	.394	.413	.428	.441	.452	.462	.470	.478
	0.975	.179	.239	.278	.307	.330	.348	.364	.377	.388	.398	.406	.414
40	0.025	1006	39.4	14.0	8.41	6.17	5.01	4.30	3.84	3.50	3.25	3.06	2.90
	0.050	251	19.4	8.59	5.71	4.46	3.77	3.34	3.04	2.82	2.66	2.53	2.42
	0.950	.244	.309	.352	.383	.408	.428	.444	.458	.470	.481	.490	.499
	0.975	.184	.246	.288	.319	.344	.364	.381	.395	.407	.418	.428	.437
60	0.025	1010	39.4	13.9	8.36	6.12	4.95	4.25	3.78	3.44	3.19	3.00	2.84
	0.050	252	19.4	8.57	5.68	4.43	3.74	3.30	3.00	2.78	2.62	2.49	2.38
	0.950	.249	.317	.362	.396	.422	.443	.461	.476	.490	.501	.512	.521
	0.975	.189	.254	.299	.332	.358	.380	.398	.414	.428	.440	.451	.461
80	0.025	1012	39.4	13.9	8.33	6.09	4.93	4.22	3.75	3.42	3.16	2.97	2.81
	0.050	252	19.4	8.56	5.67	4.41	3.72	3.28	2.98	2.76	2.60	2.46	2.36
	0.950	.252	.321	.367	.402	.429	.451	.470	.486	.500	.512	.523	.533
	0.975	.191	.258	.304	.338	.366	.389	.408	.424	.439	.451	.463	.473
100	0.025	1013	39.4	13.9	8.31	6.08	4.91	4.21	3.73	3.40	3.15	2.95	2.8
	0.050	253	19.4	8.55	5.66	4.40	3.71	3.27	2.97	2.75	2.58	2.45	2.35
	0.950	.254	.323	.371	.406	.433	.456	.475	.492	.506	.519	.530	.540
	0.975	.193	.261	.307	.342	.370	.394	.413	.430	.445	.458	.470	.481
120	0.025	1014	39.4	13.9	8.30	6.06	4.90	4.19	3.72	3.39	3.14	2.94	2.78
	0.050	253	19.4	8.54	5.65	4.39	3.70	3.26	2.96	2.74	2.58	2.44	2.34
	0.950	.255	.325	.373	.408	.436	.459	.479	.495	.510	.523	.535	.545
	0.975	.194	.262	.309	.345	.374	.397	.417	.434	.450	.463	.475	.486
∞	0.025	1018	39.5	13.9	8.25	6.01	4.84	4.14	3.67	3.33	3.08	2.88	2.72
	0.050	254	19.5	8.52	5.62	4.36	3.66	3.23	2.92	2.70	2.53	2.40	2.29
	0.950	.260	.333	.383	.421	.451	.476	.497	.515	.531	.546	.559	.570
	0.975	.199	.271	.320	.359	.389	.415	.437	.456	.473	.488	.501	.514

		ν_2											
ν_1	α	13	14	15	20	25	30	40	60	80	100	120	∞
1	0.025	6.41	6.29	6.2	5.87	5.68	5.56	5.42	5.28	5.21	5.17	5.15	5.02
	0.050	4.66	4.6	4.54	4.35	4.24	4.17	4.08	4.00	3.96	3.93	3.92	3.84
	0.950	.004	.004	.004	.004	.004	.004	.004	.004	.004	.004	.003	.003
	0.975	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
2	0.025	4.96	4.85	4.76	4.46	4.29	4.18	4.05	3.92	3.86	3.82	3.80	3.68
	0.050	3.80	3.73	3.68	3.49	3.38	3.31	3.23	3.15	3.11	3.08	3.07	2.99
	0.950	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051
	0.975	.025	.025	.025	.025	.025	.025	.025	.025	.025	.025	.025	.025
3	0.025	4.34	4.24	4.15	3.85	3.69	3.58	3.46	3.34	3.28	3.25	3.22	3.11
	0.050	3.41	3.34	3.28	3.09	2.99	2.92	2.83	2.75	2.71	2.69	2.68	2.60
	0.950	.114	.114	.114	.115	.115	.116	.116	.116	.116	.116	.117	.117
	0.975	.069	.070	.070	.070	.070	.071	.071	.071	.071	.071	.071	.071
4	0.025	3.99	3.89	3.80	3.51	3.35	3.25	3.12	3.00	2.95	2.91	2.89	2.78
	0.050	3.17	3.11	3.05	2.86	2.75	2.69	2.60	2.52	2.48	2.46	2.44	2.37
	0.950	.169	.170	.170	.172	.173	.174	.174	.175	.176	.176	.176	.177
	0.975	.114	.115	.115	.116	.117	.118	.118	.119	0.12	.120	.120	.121
5	0.025	3.76	3.66	3.57	3.28	3.12	3.02	2.90	2.78	2.73	2.69	2.67	2.56
	0.050	3.02	2.95	2.90	2.71	2.60	2.53	2.44	2.36	2.32	2.30	2.29	2.21
	0.950	.214	.215	.216	.219	.221	.222	.224	.225	.226	.227	.227	.229
	0.975	.154	.154	.155	.158	.159	.160	.161	.163	.164	.164	.164	.166
6	0.025	3.60	3.50	3.41	3.12	2.96	2.86	2.74	2.62	2.57	2.53	2.51	2.40
	0.050	2.91	2.84	2.79	2.59	2.49	2.42	2.33	2.25	2.21	2.19	2.17	2.09
	0.950	.251	.252	.253	.258	.260	.262	.265	.267	.268	.269	.269	.272
	0.975	.187	.188	.189	.193	.195	.197	.199	.201	.202	.203	.203	.206
7	0.025	3.48	3.38	3.29	3.00	2.84	2.74	2.62	2.50	2.45	2.41	2.39	2.28
	0.050	2.83	2.76	2.70	2.51	2.40	2.33	2.24	2.16	2.12	2.10	2.08	2.01
	0.950	.281	.283	.284	.290	.293	.296	.299	.302	.304	.305	.306	.309
	0.975	.216	.217	.218	.223	.227	.229	.232	.235	.236	.237	.238	.241
8	0.025	3.38	3.28	3.19	2.91	2.75	2.65	2.52	2.41	2.35	2.32	2.29	2.19
	0.050	2.76	2.69	2.64	2.44	2.33	2.26	2.18	2.09	2.05	2.03	2.01	1.93
	0.950	.306	.308	.310	.317	.321	.324	.328	.332	.334	.336	.337	.341
	0.975	.240	.242	.243	0.25	.254	.256	.260	.264	.266	.267	.268	.272
9	0.025	3.31	3.20	3.12	2.83	2.67	2.57	2.45	2.33	2.27	2.24	2.22	2.11
	0.050	2.71	2.64	2.58	2.39	2.28	2.21	2.12	2.04	1.99	1.97	1.95	1.88
	0.950	.328	.330	.332	.340	.345	.349	.353	.358	.361	.362	.364	.369
	0.975	.261	.263	.265	.272	.277	.280	.285	.289	.292	.293	.294	0.3
10	0.025	3.25	3.14	3.06	2.77	2.61	2.51	2.38	2.27	2.21	2.17	2.15	2.04
	0.050	2.67	2.60	2.54	2.34	2.23	2.16	2.07	1.99	1.95	1.92	1.91	1.83
	0.950	.346	.349	.351	.360	.366	.370	.375	.381	.384	.386	.387	.394
	0.975	.279	.281	.284	.292	.298	.302	.307	.312	.315	.317	.318	.324
11	0.025	3.19	3.09	3.00	2.72	2.56	2.45	2.33	2.21	2.15	2.12	2.10	1.99
	0.050	2.63	2.56	2.50	2.31	2.19	2.12	2.03	1.95	1.91	1.88	1.86	1.78
	0.950	.362	.365	.367	.377	.384	.389	.395	.401	.405	.407	.408	.415
	0.975	.294	.297	.300	0.31	.316	.320	.326	.332	.336	.338	.339	.346
12	0.025	3.15	3.05	2.96	2.67	2.51	2.41	2.28	2.16	2.11	2.07	2.05	1.94
	0.050	2.60	2.53	2.47	2.27	2.16	2.09	2.00	1.91	1.87	1.85	1.83	1.75
	0.950	.375	.379	.382	.393	.400	.405	.412	.419	.423	.425	.427	.435
	0.975	.308	.311	.314	.325	.332	.337	.344	.351	.354	.357	.358	.367

ν_1	α	ν_2 13	14	15	20	25	30	40	60	80	100	120	∞
13	0.025	3.11	3.01	2.92	2.63	2.47	2.37	2.24	2.12	2.07	2.03	2.01	1.90
	0.050	2.57	2.50	2.44	2.25	2.13	2.06	1.97	1.88	1.84	1.81	1.80	1.72
	0.950	.388	.391	.394	.406	.414	.420	.427	.435	.439	.442	.444	.453
	0.975	.321	.324	.327	.339	.347	.352	.359	.367	.371	.374	.376	.385
14	0.025	3.08	2.97	2.89	2.60	2.44	2.33	2.21	2.09	2.03	2	1.97	1.86
	0.050	2.55	2.48	2.42	2.22	2.11	2.03	1.94	1.86	1.81	1.79	1.77	1.69
	0.950	.398	.402	.406	.418	.427	.433	.441	.449	.454	.457	.459	.469
	0.975	.332	.335	.339	.351	0.36	.366	.373	.382	.387	.389	.391	.402
15	0.025	3.05	2.94	2.86	2.57	2.41	2.30	2.18	2.06	2.00	1.96	1.94	1.83
	0.050	2.53	2.46	2.40	2.20	2.08	2.01	1.92	1.83	1.79	1.76	1.75	1.66
	0.950	.408	.412	.416	.429	.438	.445	.453	.462	.467	.470	.473	.484
	0.975	.341	.345	.349	.362	.371	.378	.386	.396	.401	.404	.406	.417
20	0.025	2.94	2.84	2.75	2.46	2.3	2.19	2.06	1.94	1.88	1.84	1.82	1.70
	0.050	2.45	2.38	2.32	2.12	2.00	1.93	1.83	1.74	1.70	1.67	1.65	1.57
	0.950	.444	.449	.453	.470	.482	.490	.501	.513	.520	.524	.527	.542
	0.975	.379	.384	.388	.405	.417	.425	.437	.449	.456	.460	.463	.479
25	0.025	2.88	2.77	2.68	2.39	2.23	2.12	1.99	1.86	1.80	1.77	1.74	1.62
	0.050	2.41	2.34	2.28	2.07	1.95	1.87	1.78	1.69	1.64	1.61	1.59	1.50
	0.950	.468	.473	.478	.498	.511	.521	.534	.548	.556	.562	.565	.584
	0.975	.403	.409	.414	.434	.448	.458	.472	.487	.495	.501	.504	.524
30	0.025	2.83	2.73	2.64	2.34	2.18	2.07	1.94	1.81	1.75	1.71	1.69	1.56
	0.050	2.38	2.30	2.24	2.03	1.91	1.84	1.74	1.64	1.60	1.57	1.55	1.45
	0.950	.484	.490	.496	.517	.532	.543	.558	.574	.584	0.59	.594	.616
	0.975	.421	.427	.433	.455	.470	.482	.497	.515	.525	.531	.535	.559
40	0.025	2.78	2.67	2.58	2.28	2.11	2.00	1.87	1.74	1.67	1.64	1.61	1.48
	0.050	2.33	2.26	2.20	1.99	1.87	1.79	1.69	1.59	1.54	1.51	1.49	1.39
	0.950	.506	.513	.519	.543	.560	.573	.590	.610	.622	.629	.634	.662
	0.975	.444	.451	.458	.483	.501	.514	.533	.554	.566	.574	0.58	.610
60	0.025	2.72	2.61	2.52	2.22	2.05	1.94	1.80	1.66	1.59	1.55	1.53	1.38
	0.050	2.29	2.22	2.16	1.94	1.82	1.74	1.63	1.53	1.48	1.45	1.42	1.31
	0.950	.529	.537	.544	.572	.591	.606	.627	.651	.665	.675	.681	.719
	0.975	.469	.477	.485	.514	.535	.550	.573	0.6	.615	.625	.632	.674
80	0.025	2.69	2.58	2.49	2.19	2.01	1.90	1.76	1.62	1.55	1.51	1.48	1.33
	0.050	2.27	2.20	2.13	1.92	1.79	1.71	1.60	1.50	1.44	1.41	1.39	1.27
	0.950	.542	.550	.557	.587	.608	.624	.647	.674	.690	.701	.708	.754
	0.975	.483	.491	.499	.530	.553	.570	.595	.625	.643	.654	.663	.714
100	0.025	2.67	2.56	2.47	2.17	1.99	1.88	1.74	1.59	1.52	1.48	1.45	1.29
	0.050	2.26	2.18	2.12	1.90	1.77	1.69	1.58	1.48	1.42	1.39	1.36	1.24
	0.950	.549	.558	.565	.596	.618	.635	0.66	.689	.706	.718	.726	.779
	0.975	.491	0.5	.508	.541	.564	.583	.609	.642	.661	.674	.683	.742
120	0.025	2.65	2.55	2.46	2.15	1.98	1.86	1.72	1.58	1.50	1.46	1.43	1.26
	0.050	2.25	2.17	2.11	1.89	1.76	1.68	1.57	1.46	1.41	1.37	1.35	1.22
	0.950	.554	.563	.571	.602	.625	.643	.668	.699	.718	.730	.739	.797
	0.975	.496	.505	.514	.548	.572	.591	.619	.653	.674	.688	.698	.763
∞	0.025	2.59	2.48	2.39	2.08	1.90	1.78	1.63	1.48	1.4	1.34	1.31	1
	0.050	2.20	2.13	2.06	1.84	1.71	1.62	1.50	1.38	1.32	1.28	1.25	1
	0.950	.581	.591	.600	.636	.664	.685	.717	.758	.785	.804	.818	1
	0.975	.525	.536	.545	.585	.615	.638	.674	.720	.750	.771	.788	1