Lecture 6: Joint distributions

MVE055 / MSG810
Mathematical statistics and discrete mathematics

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Bivariate distributions

Definition

Informal: A two-dimensional or bivariate random variable (X, Y) produces a pair of random numbers.

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For discrete random variables we have the joint density (probability mass function)

$$f_{X,Y}(i,j) = \mathsf{P}(X = i, Y = j) = \mathsf{P}(X = i \text{ and } Y = j).$$

Here $f_{X,Y}(i,j) \geqslant 0$ and $\sum_{\text{all } i,j} f_{X,Y}(i,j) = 1$.

Example

Let X and Y be the number of girls, respectively boys in a randomly chosen Swedish family. The joint density function $f_{X,Y}(x, y)$ is given in the table below.

Y / X	0	1	2	3	4
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$\mathrm{P}(X = 2) = f_{XY}(2,0) + f_{X,Y}(2,1) + f_{XY}(2,2) = 0.08$

Expected values ♡

$$\mathsf{E}(h(X,Y)) = \sum_{\text{all } i,j} h(i,j) f_{X,Y}(i,j).$$

For example:

$$\mathsf{E}(X+Y]) = \sum_{\text{all } i,j} (i+j) f_{X,Y}(i,j)$$

with $h(i,j) = i + j$.

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$$E(X+Y) = \boxed{(0+0)\cdot 0.38 + (1+0)\cdot 0.17 + ... = 1.08}$$

Marginal distributions

Given a pair of discrete random variables (X, Y) with joint density $f_{X,Y}$ density for X and Y are given by

$$f_X(i) = \sum_{\text{all } j} f_{X,Y}(i, j)$$
$$f_Y(j) = \sum_{\text{all } i} f_{X,Y}(i, j).$$

and called **marginal densities** (marginal p.m.f.'s.)

Y / X	0	1	2	3	4	f_X
0	0.38	0.16	0.04	0.01	0.01	0.60
1	0.17	0.08	0.02			0.27
2	0.05	0.02	0.01			0.08
3	0.02	0.01				0.03
4	0.02					0.02
f_Y	0.64	0.27	0.07	0.01	0.01	1

Continuous bivariate random variables

For a pair of continuous random variables: a function $f_{X,Y}(x,y)$ with properties

1. $f_{X,Y}(x,y) \geqslant 0$,
2. $\int\int f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y = 1$, and
3. $\mathsf{P}(a \leqslant X \leqslant b \text{ and } c \leqslant Y \leqslant d) = \int_a^b \int_c^d f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$.

Marginal distributions

For a bivariate continuous random variable (X, Y), the probability density functions for X and Y are given by

$$f_X(x) = \int f_{X,Y}(x, y)\mathrm{d}y$$

$$f_Y(y) = \int f_{X,Y}(x, y)\mathrm{d}x$$

Expected value

For a two-dimensional random variable (X, Y), the expected values of X and Y are given by

$$\mathsf{E}(X) = \begin{cases} \sum\limits_{\text{all } i,\, j} i f_{X,Y}(i,j), & \text{for } X \text{ discrete,} \\ \int \int x f_{X,Y}(x,y) \mathrm{d}x \mathrm{d}y, & \text{for } X \text{ continuos,} \end{cases}$$

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and

$$\mathsf{E}(Y) = \begin{cases} \sum_{\text{all } i,\ j} j f_{X,Y}(i,j), & \text{for } Y \text{ discrete,} \\ \int\int y f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y, & \text{for } Y \text{ continuous.} \end{cases}$$

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Conditional distribution

The *conditional distribution of* X *given* $Y = y$ is defined by its density

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

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Independent random variables

Two random variables X and Y are called independent if their bivariate density can be written as product of the marginal densities:

$$f_{X,Y}(u,v) = f_X(u) f_Y(v).$$

There is no "samvariation", knowing X does not explain Y, etc.

Covariance

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Covariance between random variables X and Y is defined as $\mathrm{Cov}(X, Y) = \mathsf{E}[(X - \mu_X)(Y - \mu_Y)]$, where $\mu_X = \mathsf{E}(X)$ and $\mu_Y = \mathsf{E}(Y)$.

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- According to the definition,

$$\mathrm{Cov}(X, Y) = \begin{cases} \sum_{\text{all } i,\ j} (i - \mu_X)(j - \mu_Y) f_{X,Y}(i, j), & \text{discrete} \\ \int\int (x - \mu_X)(y - \mu_Y) f_{X,Y}(x, y) \mathrm{d}x \mathrm{d}y, & \text{cont.} \end{cases}$$

- Note that $\mathrm{Cov}(X, X) = \mathsf{V}(X)$.
- **If** X and Y are independent, **then** $\mathrm{Cov}(X, Y) = 0$ and $\mathsf{E}(XY) = \mathsf{E}(X)\mathsf{E}(Y)$.
- Unit??

Rules for covariance

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For two random variables X and Y, and two numbers a and b we have

$$\mathsf{V}(aX + bY) = a^2\mathsf{V}(X) + b^2\mathsf{V}(Y) + 2ab\ \mathrm{Cov}(X, Y).$$

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Examples:

$$\mathsf{V}(2X) = \mathsf{V}(X + X) = \mathsf{V}(X) + \mathsf{V}(X) + 2\ \mathrm{Cov}(X, X) = 4\mathsf{V}(X)$$

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(“Fun” thing to do: look up the law of cosines.)

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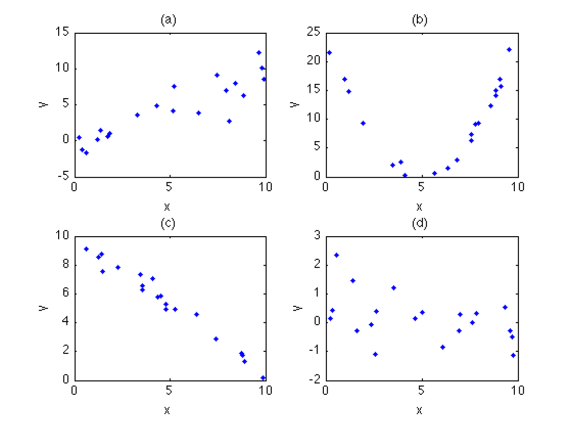
- A measure of linear relationship (linjär samvariation) of X and Y.
- It holds $-1 \leqslant \rho \leqslant 1$.
- X and Y are called uncorrelated if $\rho(X,Y) = 0$ (there is no "linjär samvariation") .
- Unit??

Visualisation



Assume 2d measurements (x_i, y_i). A scatter plot is a two-dimensional plot in which each (x_i, y_i) measurement is represented as a point in the x-y-plane.

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In the picture $r_{xy} = 0.8067$ i (a), $r_{xy} = 0.2912$ i (b), $r_{xy} = -0.9884$ i (c), och $r_{xy} = 0.3640$ i (d).

We have the following relationship between dependence and correlation:

- If X and Y are independent, then they are also uncorrelated.
- (Thus if X and Y are uncorrelated, they do not need to be independent.)

This is natural because two random variables are independent if there is no "samvariation" at all, while they are not correlated if there is no "*linjär* samvariation".

Correlation, dependence and causality

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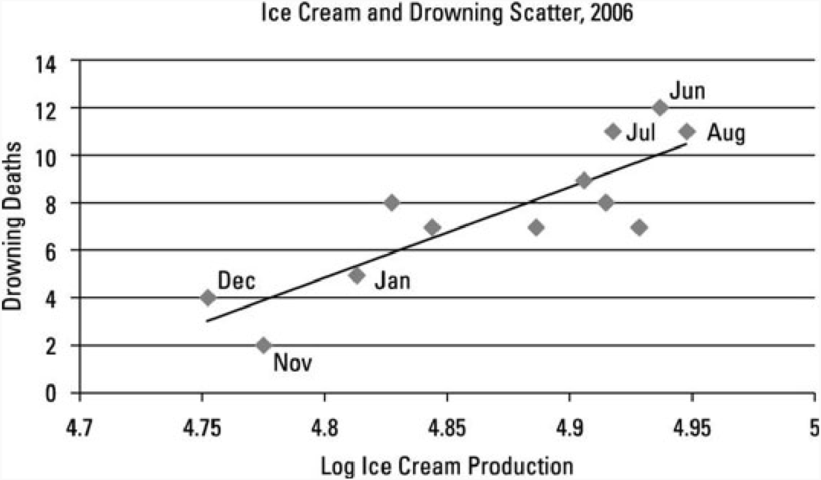
- Month with high ice cream sales tend to have more drowning accidents. Time to ban ice cream? In this example, an important variable which perhaps was not measured is the sunshine. Such variables are sometimes called confounding variables.



`https://twitter.com/dannagal/status/1244082688899919872`, September 13, 2021

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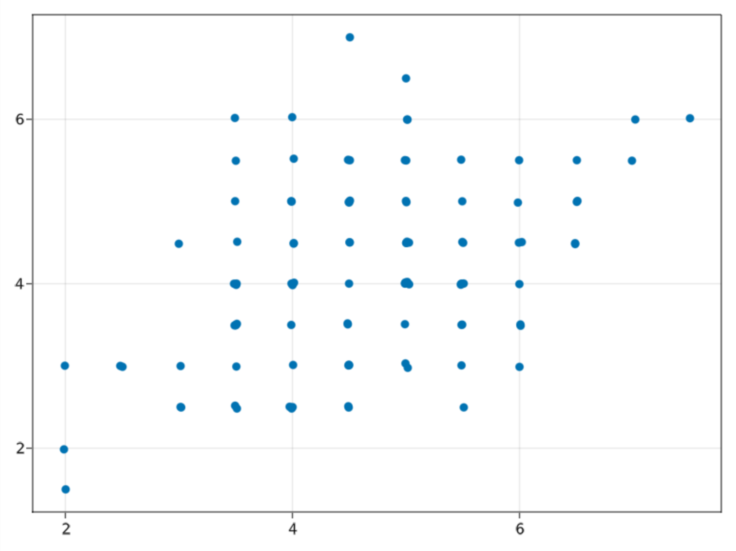
- Passing students performance on easy questions may now be negatively correlated with performance on the difficult question.

Exam points



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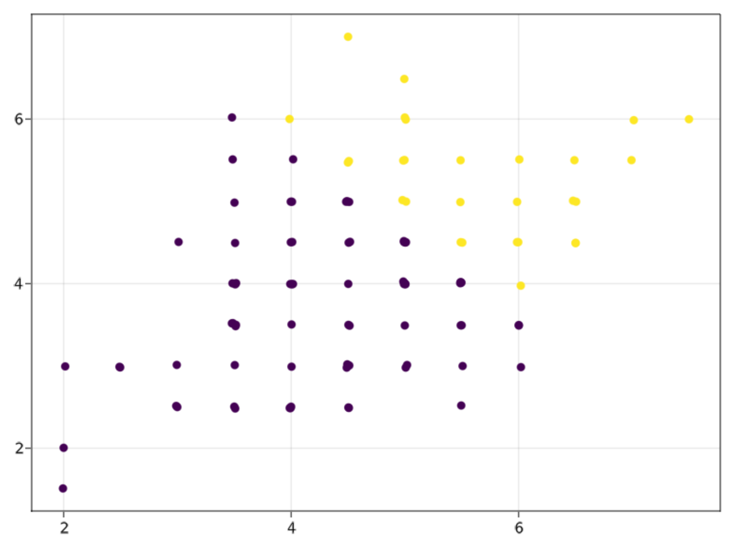

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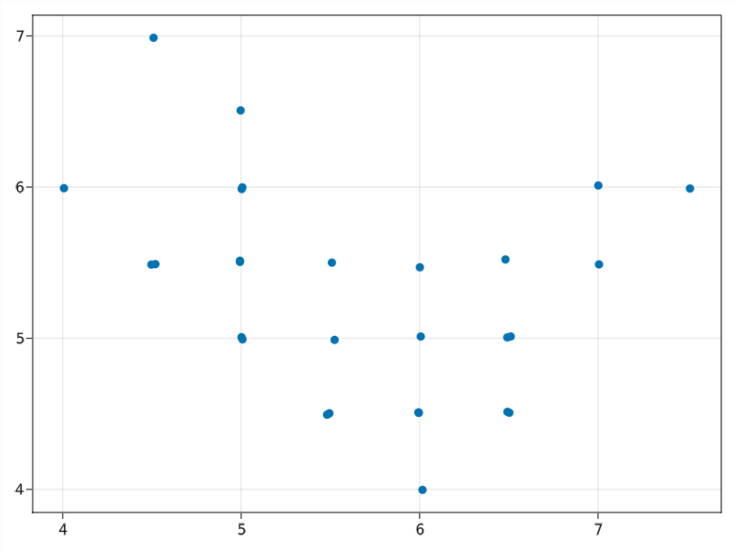

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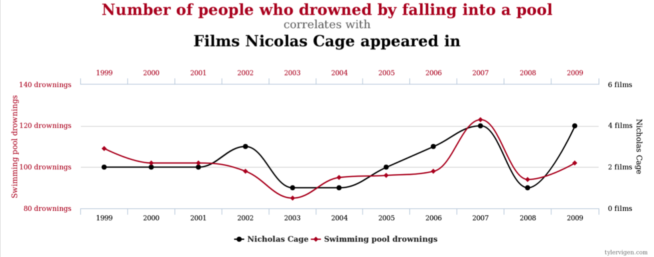
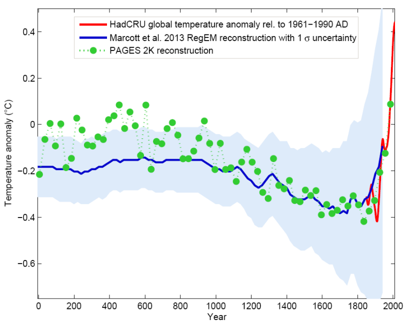
  - Candidate for a confounding variable: stress.
- We need to understand the science to answer causal questions! We will come back to this later.

Cherry picking



http://www.tylervigen.com/spurious-correlations

Thinking statistics: Global warming



Two millennia of mean surface temperatures according to different reconstructions from climate proxies with the instrumental temperature record overlaid in red.

Stefan Rahmstorf: Paleoclimate: The End of the Holocene.

`http://www.realclimate.org/index.php/archives/2013/09/paleoclimate-the-end-of-the-holocene/`.

Web. 3 Feb. 2019.