Förskjutningsmetoden

- $y' - r\cdot y = f$, $y(0)=y_0$ ($r, y_0 \in \mathbb{C}$, f Riemann-int.)

$$\Updownarrow$$

$$\frac{d}{dx}\left(y(x)e^{-rx}\right) = f(x)e^{-rx} \Leftrightarrow y(x)e^{-rx} - \underbrace{y(0)e^{-r\cdot 0}}_{=y_0} = \int_0^x f(t)e^{-rt}\,dt$$

$$\Leftrightarrow \boxed{y(x) = e^{rx}\cdot\left(y_0 + \int_0^x f(t)e^{-rt}\,dt\right)} \quad (*)$$

- $y'' + ay' + by = f$, $y(0)=y_0$, $y'(0)=y_1$ ($a, b, y_0, y_1 \in \mathbb{C}$, f Riemann-int.)

Kar. ekv: $r^2 + ar + b = 0 \rightsquigarrow$ Rötter r_1, r_2 (**obs:** $a = -(r_1+r_2)$, $b = r_1\cdot r_2$)

$$\Rightarrow y'' + ay' + by = \underbrace{\underbrace{(y' - r_1\cdot y)'}_{=z'} - r_2\underbrace{(y' - r_1 y)}_{=z}}_{y'' - \underbrace{(r_1+r_2)}_{=a}y' + \underbrace{r_1 r_2}_{=b}y} = \boxed{\begin{aligned} &z' - r_2\cdot z = f \\ &z(0) = y'(0) - r_1 y(0) \\ &= y_1 - r_1\cdot y_0 \end{aligned}} \quad (**)$$

- $z' - r_2\cdot z \overset{(**)}{=} f$, $z(0) = z_0 := y_1 - r_1\cdot y_0$

$$\overset{(*)}{\Rightarrow} \boxed{z(x) = e^{r_2 x}\left(z_0 + \int_0^x f(t)e^{-r_2 t}\,dt\right)} \quad (***)$$

- $y' - r_1\cdot y = z$, $y(0) = y_0$

$$\overset{(*)}{\Rightarrow} y(x) = e^{r_1 x}\left(y_0 + \int_0^x z(t)e^{-r_1 t}\,dt\right)$$

$$\overset{(***)}{=} e^{r_1 x}\left(y_0 + \int_0^x \left(z_0 + \int_0^t f(\tau)e^{-r_2\tau}\,d\tau\right)e^{r_2 t}\cdot e^{-r_1 t}\,dt\right)$$

$$\boxed{= e^{r_1 x}\left(y_0 + \int_0^x \left((y_1 - r_1\cdot y_0) + \int_0^t f(\tau)e^{-r_2\tau}\,d\tau\right)e^{(r_2-r_1)t}\,dt\right)}$$

$(****)$

Allmän lösningsformel!

Ex. $\lambda > 0$, $f : [0,\infty) \to \mathbb{R}$ $\int_0^\infty |f(x)|\,dx < +\infty$.
(kontinuerlig)

$$y_\lambda''(x) + \lambda^2 y_\lambda(x) = f(x), \quad y_\lambda(0) = y_\lambda'(0) = 0.$$

Beräkna $\lim_{\lambda \to \infty} y_\lambda(x)$ för alla $x \geq 0$.

(Är konvergensen likformig?)

Kar. ekv.: $r^2 + \lambda^2 = 0$ $(a = 0,\ b = \lambda^2)$ $\rightsquigarrow$ $r_1 = i\lambda$, $r_2 = -i\lambda$.

$y_0 = y_1 = 0$: $(\ast\ast\ast\ast)$ $\Rightarrow$

$$y_\lambda(x) = e^{i\lambda x} \cdot \int_0^x \underbrace{\left(\int_0^t f(z) e^{i\lambda z}\,dz \right)}_{=:F_\lambda(t)} e^{-2i\lambda t}\,dt$$

- **Obs:** $|F_\lambda(t)| \leq \int_0^t |f(z)| \cdot \underbrace{|e^{i\lambda z}|}_{=1\ \forall z}\,dz \leq \int_0^\infty |f(z)|\,dz =: M < +\infty$

$\forall t \geq 0$. oberoende av $\lambda > 0$.

- **Obs:** $\int_0^x F_\lambda(t) e^{-2i\lambda t}\,dt = \left[\frac{F_\lambda(t) \cdot e^{-2i\lambda t}}{-2i\lambda} \right]_0^x$

$$+ \frac{1}{2i\lambda} \int_0^x F_\lambda'(t) e^{-2i\lambda t}\,dt$$

$$\overset{!}{=} -\frac{1}{2i\lambda} \cdot F_\lambda(x) \cdot e^{-2i\lambda x} + \frac{1}{2i\lambda} \int_0^x f(t) e^{-i\lambda t}\,dt \quad ■$$

$F_\lambda(0) = 0$
$F_\lambda'(t) = f(t) e^{i\lambda t}$

$$\Rightarrow |y_\lambda(x)| = \underbrace{|e^{i\lambda x}|}_{=1} \cdot \underbrace{\left| \int_0^x F_\lambda(t) e^{-2i\lambda t}\,dt \right|}_{■}$$

$$|■| \leq \frac{1}{2\lambda} \underbrace{|F_\lambda(x)|}_{\leq M\ \forall x} + \frac{1}{2\lambda} \int_0^x |f(t)| \cdot \underbrace{|e^{-i\lambda t}|}_{=1}\,dt$$

$$\leq \int_0^\infty |f(t)|\,dt = M$$

$$\leq \frac{M}{\lambda} \quad \forall x \geq 0.$$

$$\Rightarrow \sup_{x \geq 0} |y_\lambda(x)| \leq \frac{M}{\lambda} \to 0, \quad \lambda \to \infty$$

$$\Rightarrow \lim_{\lambda \to \infty} y_\lambda(x) = 0 \quad (\text{likformigt på } [0,\infty)!)$$

Ex. $y''(x) + 2y'(x) + y(x) = \frac{1}{1+x^2}$, $y(0) = y'(0) = 0$.

Kar. ekv: $r^2 + 2r + 1 = 0$ $(a=2,\ b=1)$ $\rightsquigarrow$ $r_1 = r_2 = 1$

(****) $\Rightarrow$ $$\boxed{y(x) = e^{x} \int_0^x \underbrace{\left(\int_0^t \frac{e^{-z}}{1+z^2}\, dz \right)}_{=:\, g(t)} dt}$$

Oppgift: Beräkna $\lim_{x\to\infty} \frac{e^{-x}\cdot y(x)}{x}$.

$$= \frac{1}{x}\int_0^x g(t)\, dt.$$

Obs: $\lim_{t\to\infty} g(t) = \int_0^\infty \frac{e^{-z}}{1+z^2}\, dz =: \sigma \approx 0.62145\ldots$

$$= \underbrace{\frac{1}{x}\int_0^x \underbrace{(g(t)-\sigma)}_{\to 0,\ t\to\infty} dt}_{\text{VILL VISA: } \to 0,\ x\to\infty} + \sigma$$

(Fix $\varepsilon > 0$: $\exists\, t_\varepsilon$ s.a. $|g(t)-\sigma| < \varepsilon$ $\forall t > t_\varepsilon$.)

$$= \frac{1}{x}\left(\underbrace{\int_0^{t_\varepsilon} (g(t)-\sigma)\, dt}_{|\cdot| \leq \sigma\cdot t_\varepsilon} + \underbrace{\int_{t_\varepsilon}^x (g(t)-\sigma)\, dt}_{|\cdot| < \varepsilon\cdot x} \right) + \sigma$$

$$\Rightarrow \left| \frac{e^{-x}y(x)}{x} - \sigma \right| \leq \frac{\sigma\cdot t_\varepsilon}{x} + \varepsilon \quad \forall x \geq t_\varepsilon$$

$$\Rightarrow \boxed{\limsup_{x\to\infty} \left| \frac{e^{-x}y(x)}{x} - \sigma \right| \leq \varepsilon}$$

ε godtyckligt $\Rightarrow$ $$\boxed{\lim_{x\to\infty} \frac{e^{-x}y(x)}{x} = \sigma}$$

Ex. $y''(x) - 3y'(x) + 2y(x) = e^x - e^{2x}$, $y(0) = \underset{\underset{y_0}{\shortparallel}}{1}$, $y'(0) = \underset{\underset{y_1}{\shortparallel}}{0}$

Kar. ekv: $r^2 - 3r + 2 = 0 \rightsquigarrow r_1 = 1$, $r_2 = 2$

$$(****) \Rightarrow \quad y(x) = e^x\Big(1 + \int_0^x \underbrace{\Big(-1 + \underbrace{\int_0^t \underbrace{(e^{\tau} - e^{2\tau})e^{-2\tau}}_{=e^{-\tau}-1} d\tau}_{=1-e^{-t}-t}\Big)}_{=-(t+e^{-t})} e^t \, dt\Big)$$

$$= -(te^t + 1)$$

$$= e^x\Big(1 - \underbrace{\int_0^x (te^t + 1)\,dt}\Big)$$

$$= e^x(x-1) + 1 + x$$

$y(0) = 1$ OK!

$$= -e^x\big(x + e^x(x-1)\big) = \boxed{e^{2x} - xe^x - xe^{2x}.}$$

ok!

Kontroll:

$y'(0) = 2 - 1 - 1 = 0$

$$y'(x) = 2e^{2x} - (1+x)e^x - (1+2x)e^{2x}$$

$$y''(x) = 4e^{2x} - (2+x)e^x - \underbrace{\big(2 + 2(1+2x)\big)}_{=4(1+x)} e^{2x}$$

$$\Rightarrow \quad y''(x) - 3y'(x) + 2y(x) = 4e^{2x} - (2+x)e^x - 4(1+x)e^{2x}$$

$$-6e^{2x} + 3(1+x)e^x + 3(1+2x)e^{2x} + 2\big(e^{2x} - xe^x - xe^{2x}\big)$$

$$= \underbrace{(4 - 4 - 6 + 3 + 2)}_{=-1} e^{2x} + \underbrace{(-2 + 3)}_{=1} e^x + \underbrace{(-1 + 3 - 2)}_{=0} xe^x$$

$$+ \underbrace{(-4 + 6 - 2)}_{=0} xe^{2x} = \boxed{e^x - e^{2x}}$$ ok!

Ex. $y''' + ay'' + by' + cy = f$, $y(0)=y_0$, $y'(0)=y_1$, $y''(0)=y_2$

Kar.ekv: $r^3 + ar^2 + br + c = 0$ ⇝ Rötter r_1, r_2, r_3

$$\left(\boxed{\begin{aligned} a &= -(r_1+r_2+r_3) \\ b &= r_1r_2 + r_1r_3 + r_2r_3 \\ c &= -r_1r_2r_3 \end{aligned}} \right.$$

$\Rightarrow$ $y''' + ay'' + by' + cy =$

$$\underbrace{\left((y'-r_3y)' - r_2(y'-r_3y)\right)'}_{=z'} - r_1 \underbrace{\left(\underbrace{(y'-r_3y)'}_{=w'} - r_2\underbrace{(y'-r_3y)}_{=w}\right)}_{=z}$$

$$= y''' \underbrace{- (r_1+r_2+r_3)}_{=a} y'' + \underbrace{(r_1r_2+r_1r_3+r_2r_3)}_{=b} y' \underbrace{- r_1r_2r_3}_{=c} y = f$$

- $z' - r_1 z = f$, $z(0) = y''(0) - (r_2+r_3)y'(0) + r_2r_3y(0)$

$$\boxed{= y_2 - (r_2+r_3)y_1 - r_2r_3y_0 =: z_0}$$

$$\Rightarrow \boxed{z(x) = e^{r_1x}\left(z_0 + \int_0^x f(t)e^{-r_1t}\,dt\right)}$$

- $w' - r_2 w = z$, $w(0) = \boxed{y'(0) - r_3y(0) = y_1 - r_3y_0 =: w_0}$

$$\Rightarrow w(x) = e^{r_2x}\left(w_0 + \int_0^x z(t)e^{-r_2t}\,dt\right)$$

$$\boxed{= e^{r_2x}\left(w_0 + \int_0^x \left(z_0 + \int_0^t f(\tau)e^{-r_1\tau}\,d\tau\right) e^{(r_1-r_2)t}\,dt\right)}$$

- $y' - r_3 y = w$, $y(0) = y_0$

$$\Rightarrow y(x) = e^{r_3x}\left(y_0 + \int_0^x w(t)e^{-r_3t}\,dt\right)$$

$$\boxed{= e^{r_3x}\left(y_0 + \int_0^x \left(w_0 + \int_0^t \left(z_0 + \int_0^\tau f(\eta)e^{-r_1\eta}\,d\eta\right) e^{(r_1-r_2)\tau}\,d\tau\right) \cdot e^{(r_2-r_3)t}\,dt\right)}$$

Allmän lösningsformel!

$$\boxed{\begin{aligned} z_0 &= y_2 - (r_2+r_3)y_1 - r_2r_3y_0 \\ w_0 &= y_1 - r_3y_0 \end{aligned}}$$