Lösningsförslag (TMA976): Tenta 1 (14 jan-22)

1. $y=u\cdot e^{2x} \rightsquigarrow y'=(u'+2u)e^{2x}$, $y''=(u''+4u'+4u)\cdot e^{2x}$, $u(0)=1$, $u(1)=0$.

$\rightsquigarrow y''-4y'+4y=u''\cdot e^{2x}=x\cdot e^{2x} \rightsquigarrow u(x)=x^3/6+Ax+B \rightsquigarrow y(x)=\frac{1}{6}(x^3-7x+6)e^{2x}$.

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2. $\frac{dy}{1+y^2}=\frac{dx}{1+x^2} \rightsquigarrow \arctan y(x)=\arctan x+\pi/4 \rightsquigarrow y(x)=\tan(\arctan x+\pi/4)=\frac{x+1}{1-x}$

$\rightsquigarrow \lim_{x\to 1^-}(1-x)y(x)=\lim_{x\to 1^-} x+1=\underline{\underline{2}}$.

• definierad för $-\infty<x<1$

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3. $f_n(x)=e^{-x^2/n}$ (avtagande); $s_n=\sum_{k=1}^{n} f_n(k)$; $f_n(k+1)\le\int_k^{k+1} f_n(x)\,dx\le f_n(k)$ $\forall k$

$\rightsquigarrow s_n\ge\int_1^{n+1} e^{-x^2/n}dx=\sqrt{n}\int_{1/\sqrt{n}}^{(n+1)/\sqrt{n}} e^{-y^2}dy$, $s_n=e^{-1/n}+\sum_{k=1}^{n-1} f_n(k+1)\le e^{-1/n}+\sqrt{n}\int_{1/\sqrt{n}}^{\sqrt{n}} e^{-y^2}dy$

$$\rightsquigarrow \underbrace{2\int_{1/\sqrt{n}}^{(n+1)/\sqrt{n}} e^{-y^2}dy}_{\to\int_{-\infty}^{\infty}e^{-y^2}dy=\sqrt{\pi}}\le\frac{2s_n}{\sqrt{n}}\le\underbrace{\frac{2e^{-1/n}}{\sqrt{n}}}_{\to 0}+\underbrace{2\int_{1/\sqrt{n}}^{\sqrt{n}} e^{-y^2}dy}_{\to\int_{-\infty}^{\infty}e^{-y^2}dy=\sqrt{\pi}} \rightsquigarrow \lim_{n\to\infty}\frac{2s_n}{\sqrt{n}}=\sqrt{\pi}$$ ✕

Inklämningsprincipen!

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4. $x^2+2\cdot\ln(1+x)=x^2+2(x-x^2/2+x^3/3-x^4/4+O(x^5))=2x+2x^3/3-x^4/2+O(x^5)=:y$

$\rightsquigarrow y^3=8x^3+O(x^5)\rightsquigarrow \sin(x^2+2\cdot\ln(1+x))=2x+2x^3/3-x^4/2-\frac{8}{6}x^3+O(x^5)=2x-\frac{2x^3}{3}-\frac{x^4}{2}+O(x^5)$

• $-2\arctan x-\frac{1}{2}\sin x^4=-2(x-x^3/3+O(x^5))-\frac{1}{2}(x^4+O(x^{12}))=-2x+\frac{2x^3}{3}-\frac{1}{2}x^4+O(x^5)$

• $(1-\cos x)^2=(\cancel{1}-(\cancel{1}-x^2/2+O(x^4)))^2=\frac{x^4}{4}+O(x^6)$

$\rightsquigarrow \frac{\sin(x^2+2\cdot\ln(1+x))-2\arctan x-\frac{1}{2}\sin x^4}{(1-\cos x)^2}=\frac{-x^4+O(x^5)}{x^4/4+O(x^5)}\xrightarrow[x\to 0]{}-4$.

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5. $g(x)=\sum_{n=2}^{\infty}(n^2-3n)x^n=x^2\cdot\Big(\underbrace{\sum_{n=2}^{\infty}n^2\cdot x^{n-2}}_{=A(x)}-3\underbrace{\sum_{n=2}^{\infty}n\cdot x^{n-2}}_{=B(x)}\Big)$ – VILL: $g(1/2)=?$

$f(x)=\sum_{n=0}^{\infty}x^n=(1-x)^{-1}$, $|x|<1$: $f'(x)=\sum_{n=1}^{\infty}n\cdot x^{n-1}=1+x\cdot\underbrace{\Big(\sum_{n=2}^{\infty}n\cdot x^{n-2}\Big)}_{=B(x)}=(1-x)^{-2}$

$\rightsquigarrow B(x)=((1-x)^{-2}-1)/x\rightsquigarrow B(1/2)=\underline{\underline{6}}$. $\Big\}$ $f''(x)=\sum_{n=2}^{\infty}n(n-1)x^{n-2}=A(x)-B(x)=2(1-x)^{-3}$

$\rightsquigarrow A(1/2)=B(1/2)+16=\underline{22}\rightsquigarrow g(1/2)=\frac{1}{4}(A(1/2)-3B(1/2))=\frac{1}{4}(22-3\cdot 6)=\underline{\underline{1}}$.

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6. Fix $\varepsilon>0$: $\lim_{n\to\infty}\frac{1}{1+x^n}=1$ (likformigt på $[0,1-\varepsilon]$) $\rightsquigarrow A_n(\varepsilon)=\int_0^{1-\varepsilon}\frac{dx}{1+x^n}\to\int_0^{1-\varepsilon}1\,dx=1-\varepsilon$, $n\to\infty$

$B_n(\varepsilon)=\int_{1-\varepsilon}^{1}\frac{dx}{1+x^n}\le\int_{1-\varepsilon}^{1}1\,dx=\varepsilon$, $C_n\le\int_1^{\infty}\frac{dx}{x^n}=\Big[-\frac{x^{-(n-1)}}{n-1}\Big]_1^{\infty}=\frac{1}{n-1}\to 0$, $n\to\infty$

$\rightsquigarrow |A_n(\varepsilon)+B_n(\varepsilon)+C_n-1|\le\underbrace{|A_n(\varepsilon)-1|}_{\to 0}+\varepsilon+\underbrace{C_n}_{\to 0}$: $\limsup_{n\to\infty}\Big|\int_0^{\infty}\frac{dx}{1+x^n}-1\Big|\le\varepsilon$ godtyckligt! ✕

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7. $I_7:=\lim_{(x,y)\to(0,0)}\frac{x^2\cdot\ln(1+x)}{x^2+y^2}=\{\text{pol. koord}\}=\lim_{r\to 0}\frac{\cancel{r^2}}{\cancel{r^2}}\underbrace{\cos^2\theta\cdot\ln(1+r\cos\theta)}\to\underline{\underline{0}}$

$|\cdot|\le r$ $\forall\theta$, $\to 0$

$\Big|\underbrace{\cos^2\theta}_{\le 1}\cdot\ln(1+r\cos\theta)\Big|\le|r\cos\theta|\le r$

$\ln(1+t)\le t$ $\forall|t|<1$