Lösningsförslag: Tenta 2, 21/22

1. $z = y\cdot e^{x} \rightsquigarrow z' = (y'+y)e^{x} \rightsquigarrow z'' = (\underbrace{y''+2y'+y}_{=x\cdot e^{-x}})e^{x} = x \rightsquigarrow z(x) = x^3/6 \quad (z(0)=z'(0)=0)$

$$\rightsquigarrow I_1 = \lim_{x\to\infty}\frac{z(x)}{x^3} = \frac{1}{6}.$$

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2. $$\sum_{k=0}^{\infty}\cos(\alpha k)\,x^k \underset{-1<x<1}{=} \mathrm{Re}\left(\sum_{k=0}^{\infty}(\underbrace{e^{i\alpha}\cdot x}_{=:z,\ |z|<1})^k\right) \overset{\text{geom. sum}}{=} \mathrm{Re}\left(\frac{1}{\underbrace{1-e^{i\alpha}x}_{=z}}\right) = \mathrm{Re}\left(\frac{1-e^{-i\alpha}x}{|1-e^{i\alpha}x|^2}\right)$$

$$= \frac{1-x\cos\alpha}{1+x^2-2x\cos\alpha} \xrightarrow[(\cos\alpha\neq 1)]{x\to 1} I_2(\alpha) = \frac{1-\cos\alpha}{2(1-\cos\alpha)} = \frac{1}{2}, \quad \text{om } 0<\alpha<2\pi.$$

(existerar ej om $\alpha = 0$)

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3. $$s(k) := \sum_{q=k}^{k^2}\frac{1}{q}: \quad \underbrace{\int_k^{k^2+1}\frac{dx}{x}}_{=\ln\left(\frac{k^2+1}{k}\right)} \le s(k) \le \underbrace{1/k - 1/(k^2+1)}_{\le 1/k} + \underbrace{\int_k^{k^2+1}\frac{dx}{x}}_{=\ln\left(\frac{k^2+1}{k}\right)}$$

$$T(n) := \sum_{k=1}^{n}s(k): \quad \sum_{k=1}^{n}\underbrace{\ln\left(\frac{k^2+1}{k}\right)}_{\ge \ln k} \le T(n) \le \sum_{k=1}^{n}\underbrace{\ln\left(\frac{k^2+1}{k}\right)}_{\le \ln(k+1)} + \underbrace{\sum_{k=1}^{n}\frac{1}{k}}_{\le 1+\ln(n+1)}$$

$$\rightsquigarrow \underbrace{\int_2^{n+1}\ln x\,dx}_{} \le T(n) \le 1+\ln(n+1) + \underbrace{\int_1^{n+1}\ln x\,dx}_{=(n+1)\cdot(\ln(n+1)-1)+1}$$

$$= (n+1)\cdot(\ln(n+1)-1) - 2(\ln 2 - 1)$$

$$\rightsquigarrow \frac{T(n)}{n\cdot\ln n} \xrightarrow{n\to\infty} I_3 = 1$$

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4. $(e^{\varphi})' = \varphi'\cdot e^{\varphi}$, $(e^{\varphi})'' = (\varphi''+(\varphi')^2)e^{\varphi}$, $(e^{\varphi})''' = (\varphi'''+2\varphi''\cdot\varphi'+\varphi'(\varphi''+(\varphi')^2))e^{\varphi}$

Taylors sats: $e^{\varphi(x)} = e^{\varphi(0)} + \varphi'(0)e^{\varphi(0)}x + \frac{1}{2}(\varphi''(0)+\varphi'(0)^2)e^{\varphi(0)}x^2 + O(x^3)$

$$\rightsquigarrow \frac{e^{\varphi(x)}-(\cos x+\sin x)}{\sqrt{1+x^2}-1} = \frac{(e^{\varphi(0)}-1) + (\varphi'(0)e^{\varphi(0)}-1)x + \frac{1}{2}(\varphi''(0)+\varphi'(0)^2+1)x^2 + O(x^3)}{\cancel{1}+\frac{1}{2}x^2+O(x^4)-\cancel{1}}$$

(För att gränsvärdet skall existera: $\varphi(0)=0$, $\varphi'(0)=1$.)

$$\xrightarrow{x\to 0} I_4 = 2\cdot\frac{1}{2}\cdot(\varphi''(0)+\varphi'(0)^2+1) = \varphi''(0)+2$$

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5. $$\sum_{k=1}^{n}\frac{k}{(k+1)!} = \sum_{k=1}^{n}\left(\frac{k+1-1}{(k+1)!}\right) = \sum_{k=1}^{n}\left(\underbrace{\frac{1}{k!}}_{=b_k} - \underbrace{\frac{1}{(k+1)!}}_{=b_{k+1}}\right) \overset{\text{teleskoperande!}}{=} \underbrace{b_1}_{=1} - \underbrace{b_{n+1}}_{\to 0} \xrightarrow{n\to\infty} I_5 = 1$$

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6. Fix $\varepsilon>0$: $$\int_0^{\infty}\frac{\sin\pi x}{1+x^n}\,dx = \underbrace{\int_0^{1-\varepsilon}\frac{\sin\pi x}{1+x^n}\,dx}_{\to\int_0^{1-\varepsilon}\sin\pi x\,dx} + \underbrace{\int_{1-\varepsilon}^{1}\frac{\sin\pi x}{1+x^n}\,dx}_{|\cdot|\le\varepsilon} + \underbrace{\int_1^{\infty}\frac{\sin\pi x}{1+x^n}\,dx}_{\le\int_1^{\infty}\frac{dx}{x^n}\to 0,\ n\to\infty}$$

likf. konv. av integrand

$$\rightsquigarrow I_6 = \int_0^1\sin\pi x\,dx = \left[-\frac{1}{\pi}\cos\pi x\right]_0^1 = \frac{2}{\pi}$$

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7. Taylors sats: $$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| \le \underbrace{\frac{1}{2}\max_{|y|\le 1/2}|f''(y)|}_{<+\infty}\,|h| \to 0 \rightsquigarrow I_7(f) = f'(0) = -2$$

för alla små x,h.